All the previous worksheets are available in seewoo5.github.io/teaching/2026Spring.
Keywords: surface area, triple integral

1. What is the relation between the area of the part of the plane $x + 2y + 2z = 1$ that lies above a region D in the xy-plane, and the area of D itself?

2. Find the area of the part of the paraboloid $z = 2 - x^2 - y^2$ that lies above the plane $z = 0$.

3. Compute the triple integral

$$\int_0^1 \int_0^1 \int_0^{\sqrt{1-z^2}} \frac{z}{y+1} dxdzdy$$

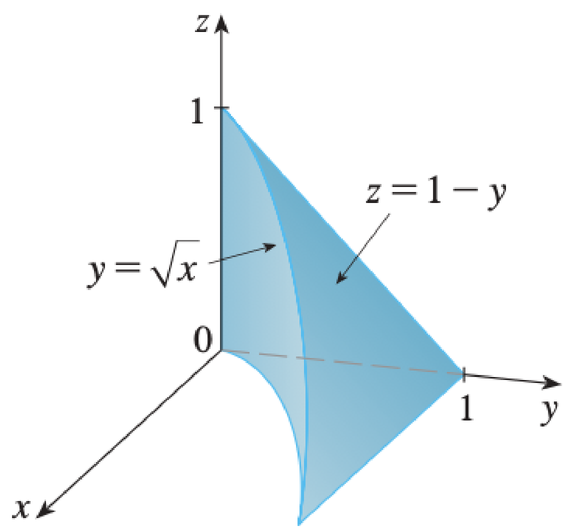
   Describe the domain of integration.

4. (a) The figure shows the region of the integration for the integral

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)dzdydx$$

   Rewrite this integral in order $dxdzdy$ and $dydzdx$.

   (b) Find the volume of the region.



1. For the plane

$$x + 2y + 2z = 1,$$

we solve for z:

$$z = f(x,y) = \frac{1 - x - 2y}{2}.$$

Then

$$f_x = -\frac{1}{2}, \qquad f_y = -1,$$

so the surface area element is

$$\sqrt{1 + f_x^2 + f_y^2}\, dA = \sqrt{1 + \frac{1}{4} + 1}\, dA = \frac{3}{2}\, dA.$$

Therefore the area of the part of the plane above a region D is

$$\iint_D \frac{3}{2}\, dA = \frac{3}{2}\,\text{Area}(D).$$

2. The paraboloid is

$$z = 2 - x^2 - y^2.$$

The condition that it lies above the plane $z = 0$ means

$$2 - x^2 - y^2 \geq 0 \quad \Longleftrightarrow \quad x^2 + y^2 \leq 2.$$

So the relevant region in the xy-plane is the disk of radius $\sqrt{2}$.
Let $f(x,y) = 2 - x^2 - y^2$. Then

$$f_x = -2x, \qquad f_y = -2y,$$

so the surface area is

$$\iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA = \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dA.$$

In polar coordinates this becomes

$$\int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{1 + 4r^2}\, r\, dr\, d\theta = 2\pi \int_0^{\sqrt{2}} r\sqrt{1 + 4r^2}\, dr.$$

Let $u = 1 + 4r^2$, so $du = 8r\, dr$. Then

$$\begin{aligned}\int_0^{\sqrt{2}} r\sqrt{1 + 4r^2}\, dr &= \frac{1}{8}\int_1^9 u^{1/2}\, du \\ &= \frac{1}{8} \cdot \frac{2}{3} u^{3/2}\Big|_1^9 = \frac{1}{12}(27 - 1) \\ &= \frac{13}{6}.\end{aligned}$$

Therefore

$$A = 2\pi \cdot \frac{13}{6} = \frac{13\pi}{3}.$$

3. The integral is

$$\int_0^1 \int_0^1 \int_0^{\sqrt{1-z^2}} \frac{z}{y+1}\, dx\, dz\, dy.$$

The domain of integration is

$$0 \leq y \leq 1, \qquad 0 \leq z \leq 1, \qquad 0 \leq x \leq \sqrt{1 - z^2}.$$

Since

$$0 \leq x \leq \sqrt{1 - z^2} \quad \Longleftrightarrow \quad x^2 + z^2 \leq 1,\ x \geq 0,\ z \geq 0,$$

the domain is the quarter-cylinder

$$\{(x,y,z) : 0 \leq y \leq 1,\ x^2 + z^2 \leq 1,\ x \geq 0,\ z \geq 0\}.$$

To compute the integral, first integrate with respect to x:

$$\int_0^{\sqrt{1-z^2}} \frac{z}{y+1}\,dx = \frac{z\sqrt{1-z^2}}{y+1}.$$

Therefore

$$\int_0^1\int_0^1\int_0^{\sqrt{1-z^2}} \frac{z}{y+1}\,dx\,dz\,dy = \int_0^1\int_0^1 \frac{z\sqrt{1-z^2}}{y+1}\,dz\,dy.$$

Since the integrand factors, we get

$$\left(\int_0^1 \frac{1}{y+1}\,dy\right)\left(\int_0^1 z\sqrt{1-z^2}\,dz\right).$$

Now

$$\int_0^1 \frac{1}{y+1}\,dy = \ln 2.$$

Also, let $u = 1-z^2$, so $du = -2z\,dz$. Then

$$\int_0^1 z\sqrt{1-z^2}\,dz = -\frac{1}{2}\int_1^0 u^{1/2}\,du = \frac{1}{2}\int_0^1 u^{1/2}\,du$$
$$= \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_0^1 = \frac{1}{3}.$$

Hence the value of the triple integral is

$$\frac{\ln 2}{3}.$$

4. Let R be the region of integration:

$$\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx.$$

This means

$$0 \le x \le 1, \qquad \sqrt{x} \le y \le 1, \qquad 0 \le z \le 1-y.$$

Since $\sqrt{x} \le y$ is equivalent to $0 \le x \le y^2$, we can also describe the region as

$$0 \le y \le 1, \qquad 0 \le x \le y^2, \qquad 0 \le z \le 1-y.$$

(a) For the order $dxdzdy$, we keep y outermost:

$$0 \le y \le 1, \qquad 0 \le z \le 1-y, \qquad 0 \le x \le y^2.$$

So

$$\int_0^1\int_0^{1-y}\int_0^{y^2} f(x,y,z)\,dx\,dz\,dy.$$

For the order $dydzdx$, we keep x outermost. Since

$$\sqrt{x} \le y \le 1-z,$$

we need $\sqrt{x} \le 1-z$, that is,

$$0 \le z \le 1-\sqrt{x}.$$

Therefore

$$0 \le x \le 1, \qquad 0 \le z \le 1-\sqrt{x}, \qquad \sqrt{x} \le y \le 1-z,$$

and the integral becomes

$$\int_0^1\int_0^{1-\sqrt{x}}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx.$$

(b) The volume is obtained by integrating 1 over the region:

$$V = \int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} 1 \, dz \, dy \, dx.$$

Integrating with respect to z gives

$$V = \int_0^1 \int_{\sqrt{x}}^1 (1-y) \, dy \, dx.$$

Then

$$\begin{aligned} \int_{\sqrt{x}}^1 (1-y) \, dy &= \left[y - \frac{y^2}{2} \right]_{\sqrt{x}}^1 \\ &= \frac{1}{2} - \sqrt{x} + \frac{x}{2}. \end{aligned}$$

Hence

$$\begin{aligned} V &= \int_0^1 \left(\frac{1}{2} - \sqrt{x} + \frac{x}{2} \right) dx \\ &= \frac{1}{2} - \frac{2}{3} + \frac{1}{4} = \frac{1}{12}. \end{aligned}$$