All the previous worksheets are available in seewoo5.github.io/teaching/2026Spring.
Keywords: cylinderical coordinate, spherical coordinate

1. Evaluate $\iiint_E z dV$, where E is enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 3$.
2. Using cylinderical coordinates, compute the following integral:

$$\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{0}^{4-x^2-y^2} \sqrt{x^2 + y^2} dz dy dx$$

3. Find the volume of the part of the ball $\rho \leq 2$ that lies between cones $\phi = \frac{\pi}{6}$ and $\phi = \frac{\pi}{3}$.
4. Express the equation of the sphere $x^2 + y^2 + (z-1)^2 = 1$ in spherical coordinates, and use it to find the volume of the sphere (the answer is $\frac{4\pi}{3}$).

1. Since the solid is enclosed by the paraboloid

$$z = x^2 + y^2 = r^2$$

and the plane $z = 3$, cylindrical coordinates are natural. The region is

$$0 \le \theta \le 2\pi, \qquad 0 \le r \le \sqrt{3}, \qquad r^2 \le z \le 3.$$

Therefore

$$\iiint_E z\,dV = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_{r^2}^{3} z\,r\,dz\,dr\,d\theta.$$

Compute:

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\sqrt{3}}\int_{r^2}^{3} z\,r\,dz\,dr\,d\theta &= \int_0^{2\pi}\int_0^{\sqrt{3}} r\left[\frac{z^2}{2}\right]_{r^2}^{3} dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{3}} \frac{r}{2}(9 - r^4)\,dr\,d\theta \\
&= \pi\int_0^{\sqrt{3}} (9r - r^5)\,dr \\
&= \pi\left[\frac{9}{2}r^2 - \frac{1}{6}r^6\right]_0^{\sqrt{3}} \\
&= \pi\left(\frac{27}{2} - \frac{9}{2}\right) = 9\pi.
\end{aligned}$$

Hence

$$\iiint_E z\,dV = 9\pi.$$

2. The bounds in Cartesian coordinates describe the region

$$-2 \le x \le 2, \qquad 0 \le y \le \sqrt{4 - x^2}, \qquad 0 \le z \le 4 - x^2 - y^2.$$

In the xy-plane this is the upper semicircle

$$x^2 + y^2 \le 4, \qquad y \ge 0,$$

so in cylindrical coordinates we have

$$0 \le \theta \le \pi, \qquad 0 \le r \le 2, \qquad 0 \le z \le 4 - r^2.$$

Also, $\sqrt{x^2 + y^2} = r$ and $dV = r\,dz\,dr\,d\theta$, so the integral becomes

$$\int_0^{\pi}\int_0^{2}\int_0^{4-r^2} r \cdot r\,dz\,dr\,d\theta = \int_0^{\pi}\int_0^{2}\int_0^{4-r^2} r^2\,dz\,dr\,d\theta.$$

Now evaluate:

$$\begin{aligned}
\int_0^{\pi}\int_0^{2}\int_0^{4-r^2} r^2\,dz\,dr\,d\theta &= \int_0^{\pi}\int_0^{2} r^2(4 - r^2)\,dr\,d\theta \\
&= \pi\int_0^{2} (4r^2 - r^4)\,dr \\
&= \pi\left[\frac{4}{3}r^3 - \frac{1}{5}r^5\right]_0^{2} \\
&= \pi\left(\frac{32}{3} - \frac{32}{5}\right) = \frac{64\pi}{15}.
\end{aligned}$$

3. The part of the ball $\rho \le 2$ lying between the cones $\phi = \frac{\pi}{6}$ and $\phi = \frac{\pi}{3}$ is described in spherical coordinates by

$$0 \le \theta \le 2\pi, \qquad \frac{\pi}{6} \le \phi \le \frac{\pi}{3}, \qquad 0 \le \rho \le 2.$$

Hence the volume is
$$\int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_0^2 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$
Evaluate:
$$\begin{aligned}\int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_0^2 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta &= \left(\int_0^{2\pi} d\theta\right)\left(\int_{\pi/6}^{\pi/3}\sin\phi\,d\phi\right)\left(\int_0^2 \rho^2\,d\rho\right)\\ &= 2\pi\left[-\cos\phi\right]_{\pi/6}^{\pi/3}\left[\frac{\rho^3}{3}\right]_0^2\\ &= 2\pi\left(\cos\frac{\pi}{6}-\cos\frac{\pi}{3}\right)\frac{8}{3}\\ &= 2\pi\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)\frac{8}{3}=\frac{8\pi(\sqrt{3}-1)}{3}.\end{aligned}$$

4. In spherical coordinates,
$$x=\rho\sin\phi\cos\theta,\qquad y=\rho\sin\phi\sin\theta,\qquad z=\rho\cos\phi.$$
Substitute these into
$$x^2+y^2+(z-1)^2=1:$$
$$\begin{aligned}\rho^2\sin^2\phi+(\rho\cos\phi-1)^2&=1\\ \rho^2\sin^2\phi+\rho^2\cos^2\phi-2\rho\cos\phi+1&=1\\ \rho^2-2\rho\cos\phi&=0.\end{aligned}$$
So the sphere is given by
$$\rho=2\cos\phi.$$
Since $\rho\geq 0$, we must have $\cos\phi\geq 0$, so
$$0\leq\phi\leq\frac{\pi}{2}.$$
Thus the ball is described by
$$0\leq\theta\leq 2\pi,\qquad 0\leq\phi\leq\frac{\pi}{2},\qquad 0\leq\rho\leq 2\cos\phi.$$
Its volume is
$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$
Compute:
$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta &= \int_0^{2\pi}\int_0^{\pi/2}\frac{(2\cos\phi)^3}{3}\sin\phi\,d\phi\,d\theta\\ &= \frac{8}{3}\int_0^{2\pi}\int_0^{\pi/2}\cos^3\phi\sin\phi\,d\phi\,d\theta\\ &= \frac{16\pi}{3}\int_0^{\pi/2}\cos^3\phi\sin\phi\,d\phi.\end{aligned}$$
Let $u=\cos\phi$, so $du=-\sin\phi\,d\phi$. Then
$$\int_0^{\pi/2}\cos^3\phi\sin\phi\,d\phi=-\int_1^0 u^3\,du=\int_0^1 u^3\,du=\frac{1}{4}.$$
Therefore the volume is
$$\frac{16\pi}{3}\cdot\frac{1}{4}=\frac{4\pi}{3}.$$