

All the previous worksheets are available in [seewoo5.github.io/teaching/2026Spring](https://seewoo5.github.io/teaching/2026Spring).

Keywords: Jacobian, vector fields, line integrals

1. Compute  $\iint_R (x - 3y) dA$  where  $R$  is the triangular region with vertices at  $(0, 0)$ ,  $(2, 1)$ ,  $(1, 2)$ , by using the change of variables  $x = 2u + v$ ,  $y = u + 2v$ .
2. Find the gradient vector field of the following functions and sketch the vector field.
  - (a)  $f(x, y) = xy$ .
  - (b)  $f(x, y) = x^2 - y^2$
3. Let  $C$  be the arc of the curve  $y = x^3$  from  $(-1, -1)$  to  $(1, 1)$ . Compute:
  - (a)  $\int_C e^x dx$ .
  - (b)  $\int_C (e^x \cos y dx - e^x \sin y dy)$ .
4. Compute the line integrals of the vector field

$$\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$$

over the following parametric curves:

- (a)  $\mathbf{r}(t) = \langle 1 - t, t \rangle$  for  $0 \leq t \leq 1$ .
- (b)  $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$  for  $0 \leq t \leq \pi/2$ .
- (c) line segment from  $(1, 0)$  to  $(1, 1)$  and then to  $(0, 1)$ .

What do you find? What is in common among the three curves?

1. The vertices  $(0, 0)$ ,  $(2, 1)$ ,  $(1, 2)$  of  $R$  corresponds to  $(0, 0)$ ,  $(1, 0)$ ,  $(0, 1)$  in the  $uv$ -plane. The line passing through  $(0, 0)$  and  $(2, 1)$  is  $y = \frac{1}{2}x$ , and it corresponds to  $u + 2v = \frac{1}{2}(2u + v) \Leftrightarrow v = 0$ . Similarly, the other two line segments correspond to  $u = 0$  and  $u + v = 1$ . So the new region  $S$  of integration in the  $uv$ -plane is

$$S = \{(u, v) : u \geq 0, v \geq 0, u + v \leq 1\}.$$

Since the Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3,$$

the integral becomes

$$\begin{aligned} \iint_R (x - 3y) dA &= \iint_S 3(-u - 5v) du dv \\ &= 3 \int_0^1 \int_0^{1-u} (-u - 5v) dv du \\ &= -3. \end{aligned}$$

- 2.

$$\nabla(xy) = \langle y, x \rangle, \quad \nabla(x^2 - y^2) = \langle 2x, -2y \rangle.$$

Sketches:



3. Parametrize  $C$  by

$$\mathbf{r}(t) = \langle t, t^3 \rangle, \quad -1 \leq t \leq 1.$$

Then  $x = t$ ,  $y = t^3$ ,  $dx = dt$ , and  $dy = 3t^2 dt$ .

$$(a) \int_C e^x dx = \int_{-1}^1 e^t dt = e - e^{-1}.$$

$$(b) \int_C (e^x \cos y dx - e^x \sin y dy) = \int_{-1}^1 e^t (\cos(t^3) - 3t^2 \sin(t^3)) dt = [e^t \cos(t^3)]_{-1}^1 = (e - e^{-1}) \cos 1.$$

$$4. (a) \mathbf{r}(t) = \langle 1 - t, t \rangle: \int_0^1 \langle t, 1 - t \rangle \cdot \langle -1, 1 \rangle dt = \int_0^1 (1 - 2t) dt = 0.$$

$$(b) \mathbf{r}(t) = \langle \cos t, \sin t \rangle: \int_0^{\pi/2} \langle \sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle dt = \int_0^{\pi/2} \cos(2t) dt = 0.$$

$$(c) \text{ Two segments: } \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 1 dt = 1, \quad \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (-1) dt = -1, \text{ so total } 0.$$

All integrals are the same. The main reason is that  $\mathbf{F}$  is a conservative vector field, i.e.,  $\mathbf{F} = \nabla f$  for  $f(x, y) = xy$ , and all the curves have the same endpoints, so the line integrals are the same by the fundamental theorem of line integrals (which you'll learn very soon).