All the previous worksheets are available in seewoo5.github.io/teaching/2026Spring.
Keywords: vector fields, line integrals, conservative fields

1. Compute the line integral
$$\int_C (x+y)\,ds,$$
where C is the line segment from $(0,0)$ to $(2,2)$.

2. (a) Show that the vector field
$$\mathbf{F}(x,y) = ye^x\mathbf{i} + (e^x + e^y)\mathbf{j}$$
is conservative. Find a potential function f.

   (b) Compute the line integral of $\mathbf{F}$ along the line segment from $(0,0)$ to $(1,2)$. Can you compute the line integral directly?

3. Explain that the vector field
$$\mathbf{F}(x,y) = xy^2\mathbf{i} - x^2y\mathbf{j}$$
is not conservative, by

   (a) computing partial derivatives of the components,

   (b) finding two different paths from $(1,0)$ to $(0,1)$ and computing the line integrals along these paths,

   (c) computing the line integral along the boundary of the unit square, oriented counterclockwise:
$$(0,0) \to (1,0) \to (1,1) \to (0,1) \to (0,0).$$

1. Parametrize the segment by
$$\mathbf{r}(t) = \langle 2t, 2t\rangle, \qquad 0 \le t \le 1.$$
Then $x + y = 4t$, and
$$ds = \|\mathbf{r}'(t)\| dt = \sqrt{2^2 + 2^2}\, dt = 2\sqrt{2}\, dt.$$
Therefore
$$\int_C (x+y)\, ds = \int_0^1 4t(2\sqrt{2})\, dt = 8\sqrt{2}\int_0^1 t\, dt = 4\sqrt{2}.$$

2. (a) Let $P(x,y) = ye^x$ and $Q(x,y) = e^x + e^y$. Then
$$\frac{\partial P}{\partial y} = e^x = \frac{\partial Q}{\partial x}$$
on all of $\mathbb{R}^2$, so $\mathbf{F}$ is conservative.
Find f from $f_x = ye^x$:
$$f(x,y) = ye^x + g(y).$$
Then
$$f_y = e^x + g'(y) = e^x + e^y \quad \Rightarrow \quad g'(y) = e^y,$$
so $g(y) = e^y + C$. A potential is
$$f(x,y) = ye^x + e^y.$$

(b) By the fundamental theorem of line integrals,
$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(1,2) - f(0,0) = (2e + e^2) - (1) = e^2 + 2e - 1.$$
Direct check on the line segment:
$$\mathbf{r}(t) = \langle t, 2t\rangle, \quad 0 \le t \le 1, \quad \mathbf{r}'(t) = \langle 1, 2\rangle.$$
Then
$$\mathbf{F}(\mathbf{r}(t)) = \langle 2te^t, e^t + e^{2t}\rangle,$$
so
$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \left(2te^t + 2e^t + 2e^{2t}\right)\, dt \\ &= \int_0^1 2(t+1)e^t\, dt + \int_0^1 2e^{2t}\, dt \\ &= 2e + (e^2 - 1) = e^2 + 2e - 1.\end{aligned}$$

3. Let $P(x,y) = xy^2$, $Q(x,y) = -x^2y$.

(a)
$$\frac{\partial P}{\partial y} = 2xy, \qquad \frac{\partial Q}{\partial x} = -2xy.$$
These are not equal in general, so $\mathbf{F}$ is not conservative.

(b) Two paths from $(1,0)$ to $(0,1)$:
Path C_1: straight line $\mathbf{r}_1(t) = \langle 1-t, t\rangle,\ 0 \le t \le 1$. Then $dx = -dt,\ dy = dt$, and
$$\begin{aligned}\int_{C_1} \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \left(xy^2\, dx + (-x^2y)\, dy\right) \\ &= \int_0^1 \left(-(1-t)t^2 - (1-t)^2 t\right)\, dt \\ &= -\int_0^1 t(1-t)\, dt = -\frac{1}{6}.\end{aligned}$$

Path C_2: two segments $(1,0) \to (0,0) \to (0,1)$. On the first segment, $y = 0$, so $\mathbf{F} = \mathbf{0}$; on the second, $x = 0$, so $\mathbf{F} = \mathbf{0}$. Hence

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0.$$

Since $-\frac{1}{6} \neq 0$, the line integral depends on path.

(c) Let $C = C_1 \cup C_2 \cup C_3 \cup C_4$, where

$$C_1 : (0,0) \to (1,0), \quad C_2 : (1,0) \to (1,1), \quad C_3 : (1,1) \to (0,1), \quad C_4 : (0,1) \to (0,0).$$

Then

$$\oint_C (xy^2\,dx - x^2y\,dy) = \sum_{k=1}^{4} \int_{C_k} (xy^2\,dx - x^2y\,dy).$$

On C_1, $y = 0$, so the integrand is 0: $\int_{C_1} = 0$.
On C_2, $x = 1$, $y = t$, $0 \le t \le 1$, $dx = 0$, $dy = dt$:

$$\int_{C_2} (xy^2\,dx - x^2y\,dy) = \int_0^1 (-t)\,dt = -\frac{1}{2}.$$

On C_3, $y = 1$, $x = 1 - t$, $0 \le t \le 1$, $dx = -dt$, $dy = 0$:

$$\int_{C_3} (xy^2\,dx - x^2y\,dy) = \int_0^1 (1-t)(-1)\,dt = -\frac{1}{2}.$$

On C_4, $x = 0$, so the integrand is 0: $\int_{C_4} = 0$.
Therefore

$$\oint_C (xy^2\,dx - x^2y\,dy) = 0 - \frac{1}{2} - \frac{1}{2} + 0 = -1 \neq 0.$$