All the previous worksheets are available in seewoo5.github.io/teaching/2026Spring.
Keywords: find potential function, curl and divergence

1. Show that the vector field
$$\mathbf{F}(x,y) = (2xy + 3y^2)\mathbf{i} + (x^2 + 6xy + 4)\mathbf{j}$$
is conservative on $\mathbb{R}^2$. Find a potential function f such that $\nabla f = \mathbf{F}$.

2. Compute the divergence and curl of each vector field. In each case, also verify directly that
$$\nabla \cdot (\nabla \times \mathbf{V}) = 0$$
for that vector field:

   (a)
   $$\mathbf{F}(x,y,z) = xyz\,\mathbf{i} + (x^2z + y^2)\,\mathbf{j} + (xy^2 + z^2)\,\mathbf{k}.$$

   (b)
   $$\mathbf{G}(x,y,z) = yze^x\,\mathbf{i} + xze^y\,\mathbf{j} + xye^z\,\mathbf{k}.$$

3. (a) Find a potential function for the vector field
   $$\mathbf{H}(x,y,z) = yz\,\mathbf{i} + (xz + e^y)\,\mathbf{j} + (xy + 2z)\,\mathbf{k}.$$

   (b) Compute the line integral
   $$\int_C \mathbf{H} \cdot d\mathbf{r},$$
   where C is the helix
   $$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle, \qquad 0 \le t \le 2\pi,$$
   from $(1,0,0)$ to $(1,0,2\pi)$.

1. Let

$$P(x,y) = 2xy + 3y^2, \qquad Q(x,y) = x^2 + 6xy + 4.$$

Then

$$\frac{\partial P}{\partial y} = 2x + 6y = \frac{\partial Q}{\partial x}$$

on all of $\mathbb{R}^2$, so $\mathbf{F}$ is conservative.

To find a potential function, integrate $f_x = P$:

$$f(x,y) = \int (2xy + 3y^2)\,dx = x^2y + 3xy^2 + g(y).$$

Differentiate with respect to y:

$$f_y = x^2 + 6xy + g'(y).$$

Since $f_y = Q = x^2 + 6xy + 4$, we get $g'(y) = 4$, so $g(y) = 4y + C$. Hence one potential function is

$$f(x,y) = x^2y + 3xy^2 + 4y.$$

2. (a) For

$$\mathbf{F}(x,y,z) = \langle xyz, x^2z + y^2, xy^2 + z^2\rangle,$$

the divergence is

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(x^2z + y^2) + \frac{\partial}{\partial z}(xy^2 + z^2) = yz + 2y + 2z.$$

The curl is

$$\begin{aligned}\nabla \times \mathbf{F} &= \left\langle \frac{\partial}{\partial y}(xy^2 + z^2) - \frac{\partial}{\partial z}(x^2z + y^2),\ \frac{\partial}{\partial z}(xyz) - \frac{\partial}{\partial x}(xy^2 + z^2),\ \frac{\partial}{\partial x}(x^2z + y^2) - \frac{\partial}{\partial y}(xyz)\right\rangle \\ &= \langle 2xy - x^2,\ xy - y^2,\ xz\rangle.\end{aligned}$$

Therefore

$$\nabla \cdot (\nabla \times \mathbf{F}) = \frac{\partial}{\partial x}(2xy - x^2) + \frac{\partial}{\partial y}(xy - y^2) + \frac{\partial}{\partial z}(xz) = (2y - 2x) + (x - 2y) + x = 0.$$

(b) For

$$\mathbf{G}(x,y,z) = \langle yze^x, xze^y, xye^z\rangle,$$

the divergence is

$$\nabla \cdot \mathbf{G} = \frac{\partial}{\partial x}(yze^x) + \frac{\partial}{\partial y}(xze^y) + \frac{\partial}{\partial z}(xye^z) = yze^x + xze^y + xye^z.$$

The curl is

$$\begin{aligned}\nabla \times \mathbf{G} &= \left\langle \frac{\partial}{\partial y}(xye^z) - \frac{\partial}{\partial z}(xze^y),\ \frac{\partial}{\partial z}(yze^x) - \frac{\partial}{\partial x}(xye^z),\ \frac{\partial}{\partial x}(xze^y) - \frac{\partial}{\partial y}(yze^x)\right\rangle \\ &= \langle x(e^z - e^y),\ y(e^x - e^z),\ z(e^y - e^x)\rangle.\end{aligned}$$

Therefore

$$\nabla \cdot (\nabla \times \mathbf{G}) = \frac{\partial}{\partial x}\big(x(e^z - e^y)\big) + \frac{\partial}{\partial y}\big(y(e^x - e^z)\big) + \frac{\partial}{\partial z}\big(z(e^y - e^x)\big) = (e^z - e^y) + (e^x - e^z) + (e^y - e^x) = 0.$$

3. (a) Let

$$\mathbf{H}(x,y,z) = \langle yz, xz + e^y, xy + 2z \rangle.$$

To find a potential function, start with $f_x = yz$:

$$f(x,y,z) = \int yz\,dx = xyz + g(y,z).$$

Differentiate with respect to y:

$$f_y = xz + g_y(y,z).$$

Since $f_y = xz + e^y$, we get

$$g_y(y,z) = e^y,$$

so

$$g(y,z) = e^y + h(z).$$

Then

$$f_z = xy + h'(z).$$

Since $f_z = xy + 2z$, we get $h'(z) = 2z$, hence $h(z) = z^2 + C$. Therefore one potential function is

$$f(x,y,z) = xyz + e^y + z^2.$$

Since

$$\nabla f = \langle yz, xz + e^y, xy + 2z \rangle = \mathbf{H},$$

this shows that $\mathbf{H}$ is conservative.

(b) Since $\mathbf{H} = \nabla f$, the fundamental theorem of line integrals gives

$$\int_C \mathbf{H} \cdot d\mathbf{r} = f(1,0,2\pi) - f(1,0,0).$$

Using $f(x,y,z) = xyz + e^y + z^2$,

$$f(1,0,2\pi) = 1 + 4\pi^2, \qquad f(1,0,0) = 1.$$

Therefore

$$\int_C \mathbf{H} \cdot d\mathbf{r} = 4\pi^2.$$