

All the previous worksheets are available in seewoo5.github.io/teaching/2026Spring.

Keywords: Green's theorem

1. Let C be a triangle with vertices at $(0, 0)$, $(1, 0)$, and $(0, 1)$, oriented counterclockwise. Let $\mathbf{F}(x, y) = \langle 2xy, x^2 + x \rangle$. Compute the line integral of \mathbf{F} along C , by
 - (a) directly evaluating the line integral, and
 - (b) using Green's theorem.
2. Let C be the circle centered at $(1, 2)$ with radius 3, oriented counterclockwise. Let $\mathbf{F}(x, y) = \langle -2y + ye^{xy}, 5x + xe^{xy} \rangle$. Compute the line integral of \mathbf{F} along C using Green's theorem.

3. Let

$$\mathbf{F}(x, y) = -\frac{y}{x^2 + y^2} \mathbf{i} + \frac{x}{x^2 + y^2} \mathbf{j} = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}.$$

- (a) (If you haven't done before) show that $Q_x = P_y$ for all $(x, y) \neq (0, 0)$.
- (b) In the last quiz, we saw that the line integral of \mathbf{F} along the unit circle centered at the origin is 2π . However, (a) says that $Q_x - P_y = 0$ and hence Green's theorem would suggest that the line integral along any closed curve should be 0. Explain why this does not contradict Green's theorem.
- (c) (Hard!) Let C be the square with vertices at $(2, 2)$, $(-2, 2)$, $(-2, -2)$, and $(2, -2)$, oriented counterclockwise. Compute the line integral of \mathbf{F} along C .

1. Let

$$P(x, y) = 2xy, \quad Q(x, y) = x^2 + x.$$

Directly, split C into the three sides.

$$C_1 : \mathbf{r}(t) = \langle t, 0 \rangle, \quad 0 \leq t \leq 1.$$

Then $P = 0$ and $dy = 0$, so the integral is 0.

Next,

$$C_2 : \mathbf{r}(t) = \langle 1 - t, t \rangle, \quad 0 \leq t \leq 1.$$

Here $dx = -dt$, $dy = dt$, so

$$\begin{aligned} \int_{C_2} P dx + Q dy &= \int_0^1 [-2(1-t)t + ((1-t)^2 + (1-t))] dt \\ &= \int_0^1 (2 - 5t + 3t^2) dt = \frac{1}{2}. \end{aligned}$$

Finally,

$$C_3 : \mathbf{r}(t) = \langle 0, 1 - t \rangle, \quad 0 \leq t \leq 1,$$

gives $P = Q = 0$, so the integral is 0. Therefore

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2}.$$

By Green's theorem,

$$\int_C P dx + Q dy = \iint_D (Q_x - P_y) dA.$$

Since

$$Q_x = 2x + 1, \quad P_y = 2x,$$

we have $Q_x - P_y = 1$. Hence

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D 1 dA = \text{Area}(D) = \frac{1}{2}.$$

2. Let

$$P(x, y) = -2y + ye^{xy}, \quad Q(x, y) = 5x + xe^{xy}.$$

Then

$$Q_x = 5 + e^{xy} + xye^{xy}, \quad P_y = -2 + e^{xy} + xye^{xy}.$$

Therefore

$$Q_x - P_y = 7.$$

By Green's theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D 7 dA = 7 \cdot \text{Area}(D) = 63\pi.$$

3. (a) Here

$$P(x, y) = -\frac{y}{x^2 + y^2}, \quad Q(x, y) = \frac{x}{x^2 + y^2}.$$

For $(x, y) \neq (0, 0)$,

$$Q_x = \frac{(x^2 + y^2) - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2},$$

and

$$P_y = \frac{-(x^2 + y^2) + 2y^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

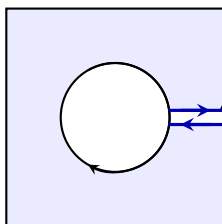
Thus $Q_x = P_y$ away from the origin.

- (b) This does not contradict Green's theorem because \mathbf{F} is not defined at $(0, 0)$. Green's theorem requires the vector field to be defined and continuously differentiable on a region containing the inside of the curve.
- (c) Let S be the square, and let D be the annular region between S and the unit circle. On D , the partial derivatives are continuous and $Q_x - P_y = 0$. Green's theorem gives

$$0 = \iint_D (Q_x - P_y) dA = \int_S \mathbf{F} \cdot d\mathbf{r} - \int_{\text{unit circle}} \mathbf{F} \cdot d\mathbf{r}.$$

The minus sign appears because the inner boundary is oriented clockwise. Since the unit circle integral is 2π ,

$$\int_S \mathbf{F} \cdot d\mathbf{r} = 2\pi.$$



This actually shows that the shape of the curve does not matter, as long as it encloses the origin once in the counterclockwise direction. In fact, if a closed curve C encloses the origin n times in the counterclockwise direction, then the line integral of \mathbf{F} along C is $2n\pi$.