All the previous worksheets are available in seewoo5.github.io/teaching/2026Spring.
Keywords: Parametric surface, Surface integral

1. Consider a surface S defined by the parametric equations

$$\mathbf{r}(u,v) = \langle u^2, uv, \frac{1}{2}v^2\rangle$$

with $0 \le u \le 2$ and $0 \le v \le 1$. Compute the surface area of S.

2. Let S be the helicoid with vector equation $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$ for $0 \le u \le 1$ and $0 \le v \le 2\pi$.

   (a) Compute the surface integral

   $$\iint_S \sqrt{x^2+y^2}\, dS$$

   (b) Let $\mathbf{F}$ be the vector field

   $$\mathbf{F}(x,y,z) = \langle x, y, z\rangle$$

   Compute the surface integral

   $$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot \mathbf{n} dS$$

   with upward orientation.

1. Let

$$\mathbf{r}(u,v) = \left\langle u^2,\ uv,\ \frac{1}{2}v^2 \right\rangle, \qquad 0 \le u \le 2,\ 0 \le v \le 1.$$

Then

$$\mathbf{r}_u = \langle 2u, v, 0\rangle, \qquad \mathbf{r}_v = \langle 0, u, v\rangle.$$

So

$$\mathbf{r}_u \times \mathbf{r}_v = \langle v^2, -2uv, 2u^2\rangle,$$

and

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{v^4 + 4u^2v^2 + 4u^4} = \sqrt{(v^2+2u^2)^2} = v^2 + 2u^2.$$

Hence the surface area is

$$A(S) = \int_0^2 \int_0^1 (v^2 + 2u^2)\, dv\, du = \int_0^2 \left(\frac{1}{3} + 2u^2\right) du = 6.$$

2. Let

$$\mathbf{r}(u,v) = \langle u\cos v,\ u\sin v,\ v\rangle, \qquad 0 \le u \le 1,\ 0 \le v \le 2\pi.$$

Then

$$\mathbf{r}_u = \langle \cos v, \sin v, 0\rangle, \qquad \mathbf{r}_v = \langle -u\sin v, u\cos v, 1\rangle,$$

so

$$\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u\rangle, \qquad |\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{1+u^2}.$$

(a) On S, we have

$$x = u\cos v, \quad y = u\sin v \quad \Longrightarrow \quad \sqrt{x^2+y^2} = u.$$

So

$$\iint_S \sqrt{x^2+y^2}\, dS = \int_0^{2\pi} \int_0^1 u\sqrt{1+u^2}\, du\, dv = 2\pi \int_0^1 u\sqrt{1+u^2}\, du.$$

Hence

$$\iint_S \sqrt{x^2+y^2}\, dS = 2\pi \left[\frac{1}{3}(1+u^2)^{3/2}\right]_0^1 = \frac{2\pi}{3}(2\sqrt{2} - 1).$$

(b) With $\mathbf{F}(x,y,z) = \langle x, y, z\rangle$,

$$\mathbf{F}(\mathbf{r}(u,v)) = \langle u\cos v,\ u\sin v,\ v\rangle.$$

So

$$\begin{aligned} \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) &= u\cos v\sin v - u\sin v\cos v + uv \\ &= uv. \end{aligned}$$

Therefore

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \int_0^1 \int_0^{2\pi} uv\, dv\, du = \int_0^1 \left(2\pi^2 u\right) du = \pi^2.$$

Since the k-component of $\mathbf{r}_u \times \mathbf{r}_v$ is $u \ge 0$, this is the upward orientation.