All the previous worksheets are available in seewoo5.github.io/teaching/2026Spring.
Keywords: Stokes' theorem, divergence theorem

1. Let $\mathbf{F}(x,y,z) = \langle x^2yz, yz^2, z^3e^{xy}\rangle$ and S be the part of the sphere $x^2+y^2+z^2=5$ that lies above the plane $z=1$, oriented upward. Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$

2. (a) Let $\mathbf{F}(x,y,z) = \langle x,y,z\rangle$ and S be the unit sphere centered at the origin. Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$ directly.

   (b) Compute the same integral using the divergence theorem.

3. Let $\mathbf{F}(x,y,z) = \langle y+e^x,\ -x+\sin y,\ z+e^z\rangle$, and let C be the boundary of the triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, oriented counterclockwise as viewed from above. Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ using Stokes' theorem.

4. Let $\mathbf{F}(x,y,z) = \langle x+e^y,\ y+\sin z,\ z^2+\cos x\rangle$, and let S be the closed surface of the cylinder $x^2+y^2 \le 1$, $0 \le z \le 2$, oriented outward. Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$ using the divergence theorem.

1. By Stokes' theorem,
$$\iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r},$$
where C is the boundary circle $x^2+y^2=4,\ z=1$, oriented counterclockwise from above. On C, $\mathbf{F}=\langle x^2y,\ y,\ e^{xy}\rangle$. C can be parametrized by $\mathbf{r}(t)=\langle 2\cos t, 2\sin t, 1\rangle$ for $0\le t\le 2\pi$, and $\mathbf{t}'(t)=\langle -2\sin t, 2\cos t, 0\rangle$. Thus the line integral can be computed as
$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt\\ &= \int_0^{2\pi} \langle 8\cos^2 t\sin t, 2\sin t, 2e^{2\cos t\sin t}\rangle\cdot\langle -2\sin t, 2\cos t, 0\rangle dt\\ &= \int_0^{2\pi}(-16\cos^2 t\sin^2 t + 4\sin t\cos t)dt\\ &= \int_0^{2\pi}(-4\sin^2(2t)+2\sin(2t))dt\\ &= \int_0^{2\pi}(2\cos(4t)-2+2\sin(2t))dt\\ &= \left[\frac{1}{2}\sin(4t)-2t-\cos(2t)\right]_0^{2\pi}\\ &= -4\pi.\end{aligned}$$
Here we used the double angle formulas $\sin(2t)=2\sin t\cos t$ and $\sin^2(2t)=\frac{1-\cos(4t)}{2}$.
It is actually easier to compute the line integral using Stokes' theorem again (or Green's theorem in some sense). You can write the line integral as a surface integral again, but over the disk $D:\ x^2+y^2\le 4,\ z=1$ instead of S. Since $\mathbf{n}=\langle 0,0,1\rangle$ on the disk, we have
$$\begin{aligned}\iint_D (\nabla\times\mathbf{F})\cdot d\mathbf{S} &= \iint_D (\nabla\times\mathbf{F})\cdot\mathbf{n}dS\\ &= \iint_D \langle\cdots,\cdots,-x^2\rangle\cdot\langle 0,0,1\rangle dS\\ &= \iint_D -x^2\,dS\\ &= -\int_0^{2\pi}\int_0^2 r^2\cos^2\theta\cdot r dr d\theta\\ &= -4\pi.\end{aligned}$$
The second last equality uses the polar coordinate.

2. For $\mathbf{F}=\langle x,y,z\rangle$ on the unit sphere $x^2+y^2+z^2=1$:

   (a) On the unit sphere, $\mathbf{n}=\langle x,y,z\rangle$, so
   $$\mathbf{F}\cdot\mathbf{n}=x^2+y^2+z^2=1.$$
   Therefore
   $$\iint_S \mathbf{F}\cdot d\mathbf{S}=\iint_S 1\,dS=\text{Area}(S)=4\pi.$$

   (b) By the divergence theorem,
   $$\iint_S \mathbf{F}\cdot d\mathbf{S}=\iiint_B (\nabla\cdot\mathbf{F})\,dV,$$
   where B is the unit ball. Since
   $$\nabla\cdot\mathbf{F}=1+1+1=3,$$

we get

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 3 \cdot \mathrm{Vol}(B) = 3 \cdot \frac{4\pi}{3} = 4\pi.$$

3. Let S be the triangular face in the plane $x + y + z = 1$ with upward orientation and boundary C. By Stokes' theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

For $\mathbf{F} = \langle y + e^x,\ -x + \sin y,\ z + e^z \rangle$,

$$\nabla \times \mathbf{F} = \left\langle 0, 0, \frac{\partial(-x + \sin y)}{\partial x} - \frac{\partial(y + e^x)}{\partial y} \right\rangle = \langle 0, 0, -2 \rangle.$$

So

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = -2 \iint_S n_z \, dS = -2 \iint_D dA,$$

where D is the projection onto the xy-plane: $x \geq 0,\ y \geq 0,\ x + y \leq 1$. Its area is $\frac{1}{2}$, hence

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = -2 \cdot \frac{1}{2} = -1.$$

4. By the divergence theorem,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E (\nabla \cdot \mathbf{F}) \, dV,$$

where $E = \{(x, y, z) : x^2 + y^2 \leq 1,\ 0 \leq z \leq 2\}$.

For $\mathbf{F} = \langle x + e^y,\ y + \sin z,\ z^2 + \cos x \rangle$,

$$\nabla \cdot \mathbf{F} = 1 + 1 + 2z = 2 + 2z.$$

In cylindrical coordinates,

$$\iiint_E (2 + 2z) \, dV = \int_0^{2\pi} \int_0^1 \int_0^2 (2 + 2z) \, r \, dz \, dr \, d\theta.$$

Compute:

$$\int_0^2 (2 + 2z) \, dz = 8, \quad \int_0^1 r \, dr = \frac{1}{2}, \quad \int_0^{2\pi} d\theta = 2\pi.$$

Therefore

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 8 \cdot \frac{1}{2} \cdot 2\pi = 8\pi.$$