Using Machine Learning for Number Theory

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September 8, 2025. Math254A guest lecture

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- Tell you about the potential usefulness of machine learning in number theory (but without AI hype)
- Two use cases: Predicting the rank of elliptic curves and Galois groups of number fields
- Give you some ideas for your future research

Elliptic Curve

Definition

An **elliptic curve** over a field K is a smooth, projective, algebraic curve of genus one, with a specified point $O \in E(K)$.

More concretely, it can be given by (the projectivization of) a Weierstrass equation of the form

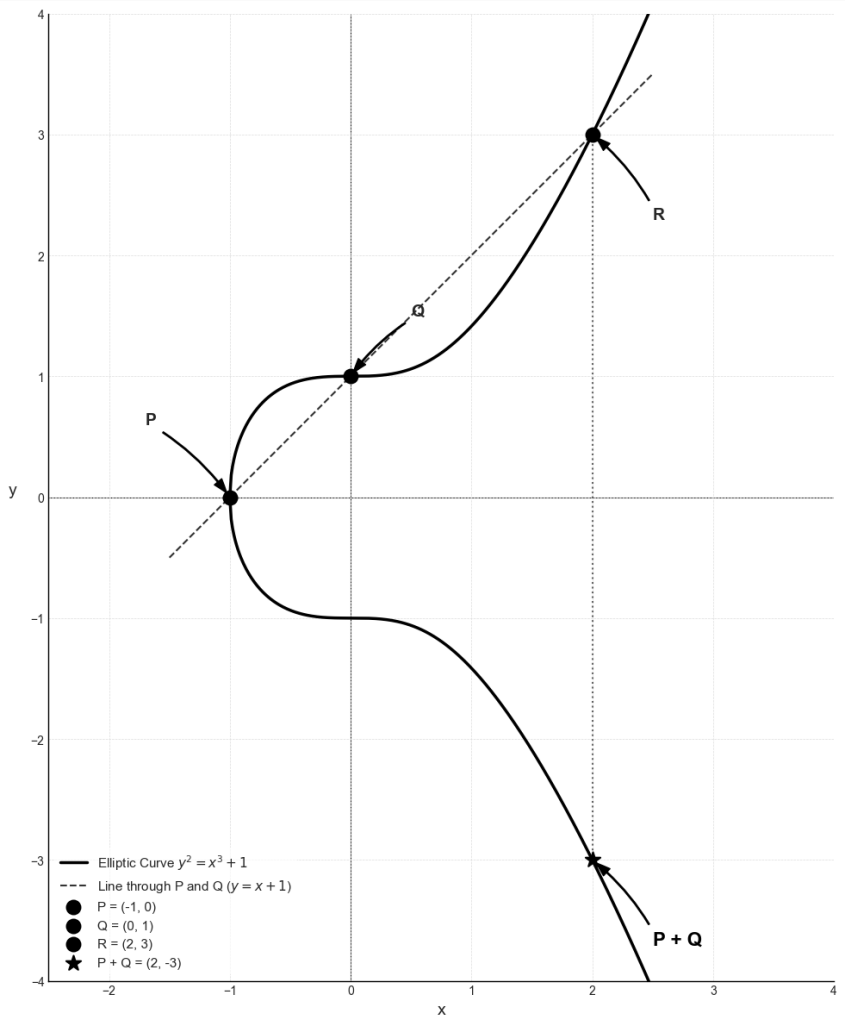
$$y^2 = x^3 + ax + b$$

where $a, b \in K$ and the curve is nonsingular[1].

[1] $\Delta = -16(4a^3 + 27b^2) \neq 0$

Group Structure of $E(K)$

Elliptic curves have a group structure. The point at infinity O serves as the identity element.



Mordell–Weil theorem

Theorem (Mordell–Weil)

The group of rational points $E(\mathbb{Q})$ is a finitely generated abelian group. That is,

$$E(\mathbb{Q}) \cong E(\mathbb{Q})_{tors} \oplus \mathbb{Z}^r$$

where $E(\mathbb{Q})_{tors}$ is the torsion subgroup and r is the rank of the elliptic curve.

Torsion is Easy

Theorem (Nagell–Lutz)

Let $(x, y) \in E(\mathbb{Q})$ be a torsion point.

- *If $(x, y) \neq O$, then $x, y \in \mathbb{Z}$.*
- *Either $y = 0$ or y^2 divides the discriminant Δ of the curve.*

Theorem (Mazur)

The torsion subgroup $E(\mathbb{Q})_{tors}$ is isomorphic to one of the following 15 groups:

$$\mathbb{Z}/n \quad (1 \leq n \leq 10, n = 12),$$
$$\mathbb{Z}/2 \times \mathbb{Z}/2m \quad (1 \leq m \leq 4).$$

Rank is Hard

It is not just hard, but it is *very* hard! We have the following list of open problems:

Rank is Hard

It is not just hard, but it is *very* hard! We have the following list of open problems:

- Can you find an elliptic curve over $\mathbb{Q}$ with arbitrarily large rank?
- (Goldfeld's conjecture) How many elliptic curves over $\mathbb{Q}$ have rank 0, 1, 2, . . . ?
- (BSD conjecture, Parity conjecture) How is the rank of an elliptic curve over $\mathbb{Q}$ related to its L-function?

ML?

Question

Can we predict the rank of an elliptic curve over $\mathbb{Q}$ using machine learning?

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Can we predict the rank of an elliptic curve over $\mathbb{Q}$ using machine learning?

If the target is the rank r of an elliptic curve $E/\mathbb{Q}$, then what are the features? We may want to predict r from something that is easier to compute.

He–Lee–Oliver

Yanghui He, Kyu-Hwan Lee, and Thomas Oliver[2] used classical ML algorithms (not ChatGPT!) to predict the rank of elliptic curves over $\mathbb{Q}$, using *Frobenius traces* $a_p(E)$ as features:

$$a_p(E) = p + 1 - |E(\mathbb{F}_p)|$$

where $|E(\mathbb{F}_p)|$ is the number of points on the reduced curve modulo p (when E has good reduction at p).

Note that the rank (more precisely, the isogeny class of E) is determined by $a_p(E)$ for *all* p.

[2] He–Lee–Oliver, *Machine learning invariants of arithmetic curves*, 2023

He–Lee–Oliver

Consider rank $0, 1$ curves of conductor ≤ 10000. They used logistic regression:

$$\mathbb{P}[\text{rank}(E) = 1 | \{a_{p_n}\}_{n \leq 300}] \approx \sigma(\mathbf{w} \cdot \mathbf{a} + b)$$

where $\mathbf{a} = (a_2, a_3, a_5, \ldots a_{1987}) \in \mathbb{R}^{300}$,

$$\sigma(x) = \frac{1}{1 + e^{-x}},$$

and $\mathbf{w} \in \mathbb{R}^{300}$, $b \in \mathbb{R}$ are weight and bias to be optimized. It essentially tries to find the a separating hyperplane in $\mathbb{R}^N = \mathbb{R}^{300}$.

Quiz 1

What was the accuracy of the experiment?

1. 0%
2. $(0\%, 50\%]$
3. $(50\%, 75\%]$
4. $(75\%, 90\%]$
5. $(90\%, 95\%]$
6. $(95\%, 100\%)$
7. 100%

Quiz 1

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5. $(90\%, 95\%]$
6. $(95\%, 100\%)$ (99.1%)
7. 100%

Why?

Why does it work so well?

It is not surprising

The BSD conjecture claims that the number of $\mathbb{F}_p$ points and the rank r of E are related by

$$\prod_{\substack{p \leq x \\ p \nmid \Delta_E}} \frac{|E(\mathbb{F}_p)|}{p} \sim C(\log x)^r$$

for some constant $C > 0$. In other words, the rank and $\{|E(\mathbb{F}_p)|\}_p$ are positively correlated. Since $a_p(E) = p + 1 - |E(\mathbb{F}_p)|$, the rank and $\{a_p\}_p$ are negatively correlated.

Something is very surprising

Podznyakov[3] analyzed models and data, e.g. using PCA. He plotted the average of a_p for each p and plotted it as a function in p, and got the following plot:[4]

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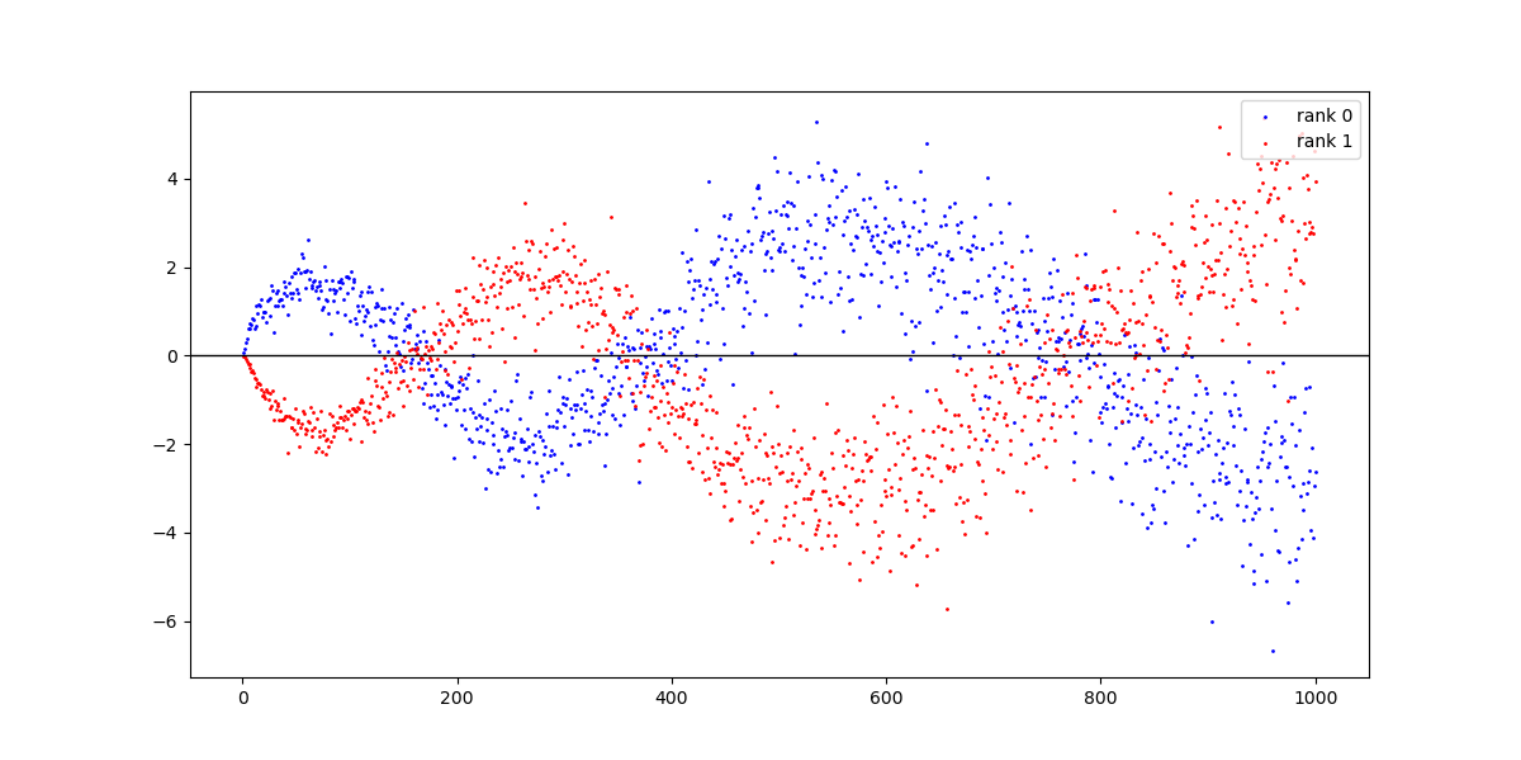

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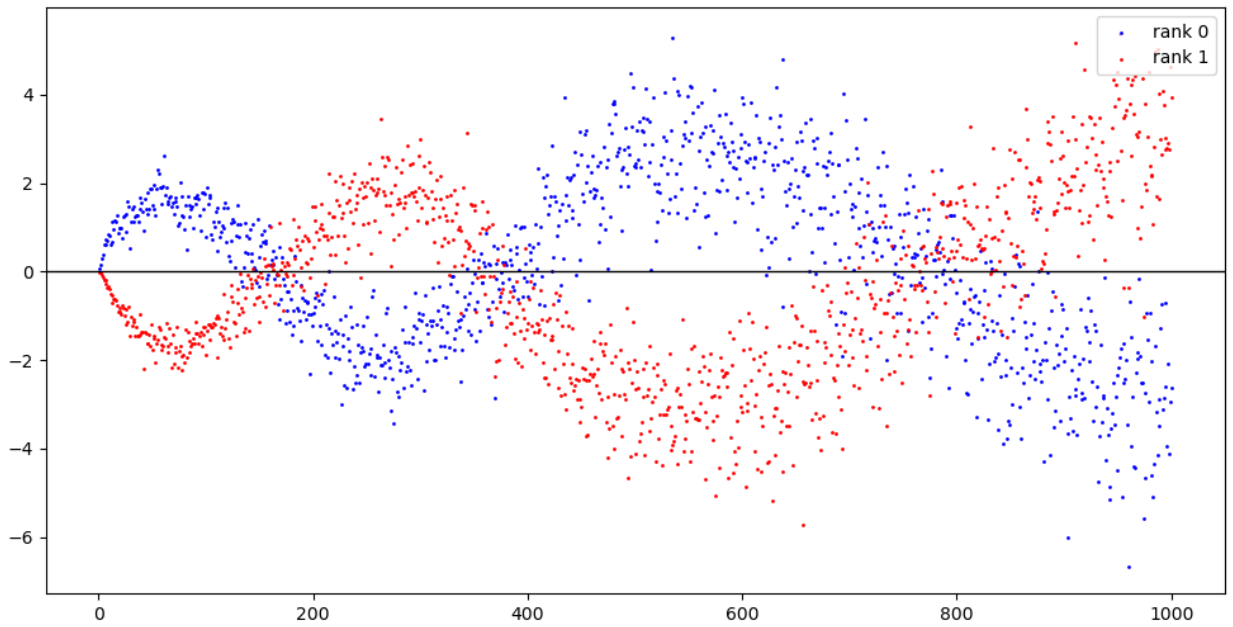

This IS surprising, since

1. We only expected a negative correlation between rank and a_p's, but this shows an oscillating pattern.
2. The equation for the "limit curve" is still unknown.

Murmuration

It is now called *murmuration*, and has become an active area of study. The oscillating pattern is observed for other "families" of L-functions, and the "limit curve" (*murmuration density*) is now known for

- Modular forms
- Maass forms
- Dirichlet characters
- Hecke characters of imaginary quadratic fields
- Elliptic curves, but in a different setup

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In this case, ML **"motivated"** mathematicians to find a new phenomenon. **However, it is incorrect to say that ML "found" new mathematics.**

Number fields

Definition

A **number field** is a finite extension K of $\mathbb{Q}$.

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$\mathbb{Q}, \mathbb{Q}(i), \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt[3]{3}), \mathbb{Q}[x]/(x^3 - 3x - 1), \ldots$

Galois group

Question

How do we compute the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ of a number field K?

It is not an easy problem.

Quiz 2-1

What is the Galois group of

$$K = \mathbb{Q}[x]/(x^4 + 2x^2 + 4)$$

1. $\mathbb{Z}/4$
2. $(\mathbb{Z}/2)^2$

Quiz 2-1

What is the Galois group of

$$K = \mathbb{Q}[x]/(x^4 + 2x^2 + 4) \simeq \mathbb{Q}(\sqrt{2}, \sqrt{-3})$$

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Quiz 2-2

What is the Galois group of

$$K = \mathbb{Q}[x]/(x^8 - x^7 + x^5 - x^4 + x^3 - x + 1)$$

1. $\mathbb{Z}/8$
2. $\mathbb{Z}/4 \times \mathbb{Z}/2$
3. $(\mathbb{Z}/2)^3$
4. D_4
5. Q_8

Quiz 2-2

What is the Galois group of

$$K = \mathbb{Q}[x]/(x^8 - x^7 + x^5 - x^4 + x^3 - x + 1) \simeq \mathbb{Q}(\zeta_{15})$$

1. $\mathbb{Z}/8$
2. $\mathbb{Z}/4 \times \mathbb{Z}/2 \simeq (\mathbb{Z}/15)^\times$
3. $(\mathbb{Z}/2)^3$
4. D_4
5. Q_8

ML?

Question

Can we use ML to predict Galois group?

Dedekind zeta function

The analogue of $a_p(E)$ for number fields is the *Dedekind zeta coefficients*.

Definition

For a number field K, the **Dedekind zeta function** $\zeta_K(s)$ is defined as

$$\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{1}{(N\mathfrak{a})^s}$$

where the sum is over all nonzero ideals $\mathfrak{a}$ of the ring of integers $\mathcal{O}_K$ of K, and $N\mathfrak{a} = |\mathcal{O}_K/\mathfrak{a}|$ is the norm of $\mathfrak{a}$.

Dedekind zeta function

It can be rewritten as

$$\zeta_K(s) = \sum_{n \geq 1} \frac{a_n(K)}{n^s}$$

for $a_n(K) = |\{\mathfrak{a} \subset \mathcal{O}_K : N\mathfrak{a} = n\}|$. These will be used as features.

Examples

$$\zeta_{\mathbb{Q}}(s) = \sum_{n \geq 1} \frac{1}{n^s} = \zeta(s)$$

$$\begin{aligned}\zeta_{\mathbb{Q}(i)}(s) &= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{4^s} + \frac{2}{5^s} + \frac{1}{8^s} + \frac{1}{9^s} + \cdots \\ &= \zeta(s)\left(\frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \cdots\right)\end{aligned}$$

Setup

We follow the setup of He–Lee–Oliver[5]

- Fix degree $\in \{4, 6, 8, 9, 10\}$.
- Input (feature): $\{a_n(K)\}_{n \leq N}$, let's say $N = 1000$.
- Output (target): Galois group
- Model: Decision Tree (= bunch of if-else statements)
- **We'll only focus on Galois (normal) extensions.**

[5] He–Lee–Oliver, *Machine Learning Number Fields*, 2022

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$$\mathbb{Z}/9, \quad (\mathbb{Z}/3)^2$$

Quiz 3-2

There are 1266 Galois nonic fields in LMFDB, 22% of them are $\mathbb{Z}/9$ and 78% of them are $(\mathbb{Z}/3)^2$. Split them randomly into train (80%) and test (20%) set. What was the accuracy of the decision tree model?

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Why? Let's look inside the tree.

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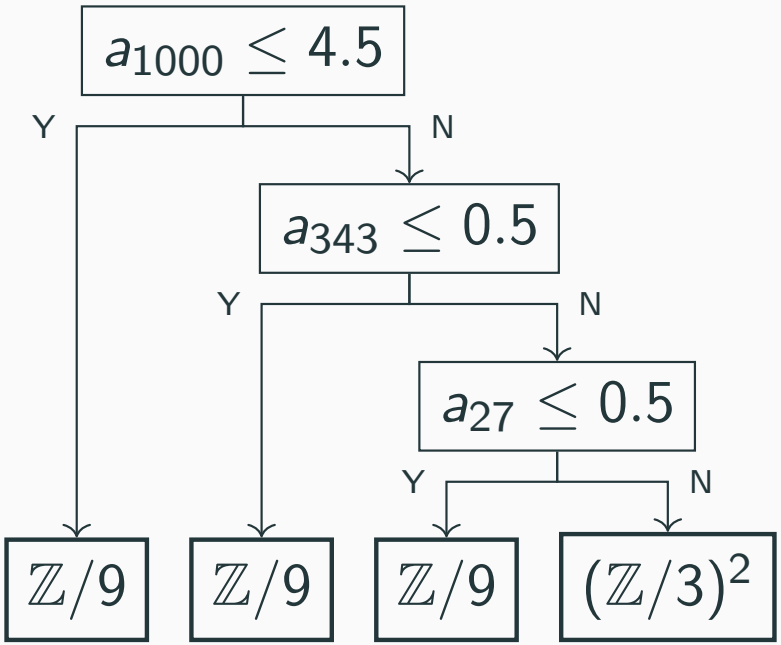

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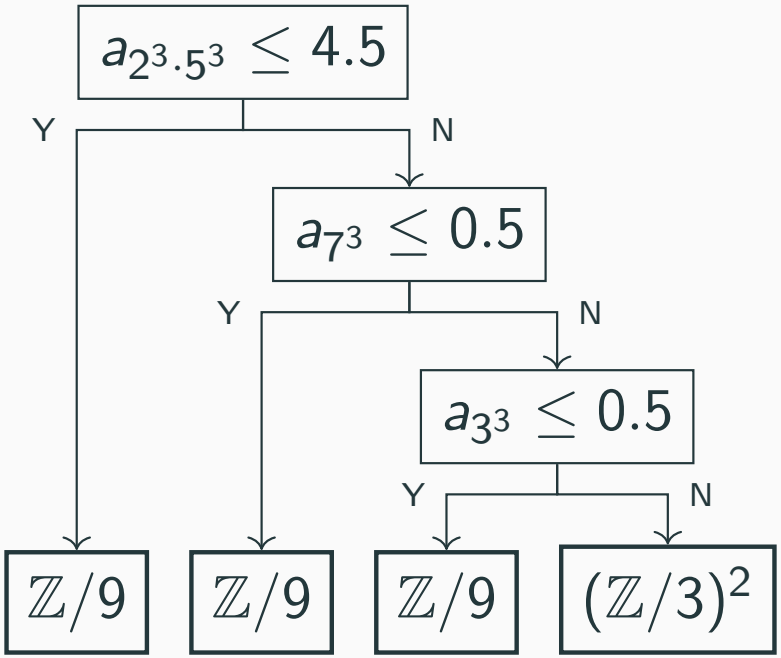

Tree's logic

All a_n's are integers, so $a_n \leq 0.5$ is equivalent to $a_n = 0$. From the two nodes below, it is natural to conjecture that

Conjecture

Let $K/\mathbb{Q}$ be a nonic Galois extension. If $a_{p^3}(K) = 0$ for some prime p, then $\mathrm{Gal}(K/\mathbb{Q}) \simeq \mathbb{Z}/9$.

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Theorem (Lee[2])

Let $K/\mathbb{Q}$ be a nonic Galois extension. Then $a_{p^3}(K) = 0$ for some prime p if and only if $\mathrm{Gal}(K/\mathbb{Q}) \simeq \mathbb{Z}/9$.

[6] Lee–Lee, *Machines Learn Number fields, But How? The Case of Galois Groups*, 2025

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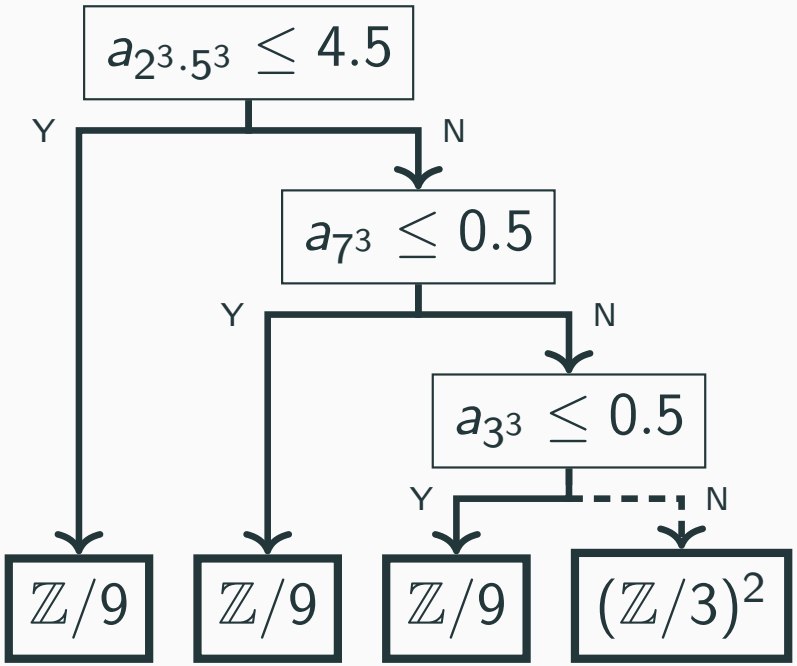
Theorem (Lee[2])

Let ℓ be a prime and $K/\mathbb{Q}$ be a degree ℓ^2 Galois extension. Then $a_{p^\ell}(K) = 0$ for some prime p if and only if $\mathrm{Gal}(K/\mathbb{Q}) \simeq \mathbb{Z}/\ell^2$.

[7] Lee–Lee, *Machines Learn Number fields, But How? The Case of Galois Groups*, 2025

Provable prediction



The bold paths *always gives a correct prediction*!

Proof

Proof

Exercise (for the end of the semester)! Use Euler factorization of Dedekind zeta function, decomposition/inertia group, etc.

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2. *For $p \equiv 1 \pmod 4$, if $a_{p^4}(K) = 1$ or $a_{p^2}(K) = 1$, then $\mathrm{Gal}(K/\mathbb{Q})$ is abelian.*
3. *If $p \equiv 3 \pmod 4$, $a_{p^4}(K) = 1$, and $a_{p^2}(K) > 0$, then $\mathrm{Gal}(K/\mathbb{Q})$ is nonabelian.*

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And the decision tree uses this as a part of its prediction logic!

Review Quiz

$$K = \mathbb{Q}[x]/(x^8 - x^7 + x^5 - x^4 + x^3 - x + 1) = \mathbb{Q}(\zeta_{15})$$

One can compute (using 'zeta_coefficients()' in SageMath) that $a_{5^2} = a_{5^4} = 1$, hence $K/\mathbb{Q}$ is abelian.

Review Quiz

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One can compute (using 'zeta_coefficients()' in SageMath) that $a_{5^2} = a_{5^4} = 1$, hence $K/\mathbb{Q}$ is abelian. In fact, this also shows $\mathrm{Gal}(K/\mathbb{Q}) \simeq \mathbb{Z}/4 \times \mathbb{Z}/2$, since $a_{p^2} = a_{p^4} = 1$ can only happen when p is totally ramified in K.

General workflow

1. Problem setup: Define target (hard to compute) and feature (easy to compute) of ML model.
2. Experiments: Start from small and simpler algorithms. If they don't work well, try larger and more complex models.
3. Interpret: If the model works much better than you expected, there should be something. Analyze the models.
4. Math: Make a conjecture from your observation, and try to prove it.

General workflow

Ideal scenarios:

- ML model works so well and is easy to interpret.
- Find something new and prove a (well-known) conjecture.
- Design a new algorithm to compute the target that is more efficient than existing algorithms.
- Find rare examples (e.g. elliptic curve of rank ≥ 30?). Probably use Reinforcement Learning.

If model works so bad

If model works so bad

...then it is also good! If you tried several models and still get a poor performance *close to random guess*, then it suggests an *equidistribution* property of the target with respect to the feature.

Unit rank of quadratic fields

For example, when you try to predict the rank of the unit group $U_K = \mathcal{O}_K^\times$ of a quadratic field $K = \mathbb{Q}(\sqrt{d})$ from the Dedekind zeta coefficients $a_n(K)$, you may get about 50% accuracy *for any model*.

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$$\mathbb{P}[a_n(K) = a \mid \mathrm{rank}(U_K) = r]$$

only depends on n and $a \in \{0, 1, 2\}$, not on $r \in \{0, 1\}$.

Possible projects

- Read the works where machine learning is used to study number theoretic objects[8]. If they only report performance but no interpretation, try to interpret the model.
- Think about your favorite number theoretic object. Can you learn any invariants of it from other invariants, using machine learning? Does LMFDB have data on it?
- There are some possible follow-up works for Lee[2], e.g. non-Galois extensions (both with or without ML).

[8] https://seewoo5.github.io/awesome-ai-for-math/