

How to talk to Aliens

Some practical tips on digesting AI-generated informal and formal proofs

Seewoo Lee, EPFL

Oct 13, 2026. AI, Combinatorics and Number Theory Seminar, Online

Disclaimer. This talk is given on October 13, 2026. Most of my “technical complaints” and some “suggestions” will likely be resolved or become obsolete in the future. But as of now, I will make this talk still useful (hopefully).

Tip

The suggestions will appear in *tip boxes* across the slides.

Prompt

The prompt I give to AI is written in *prompt boxes*.

Response

The response from AI is written in *response boxes*.

A lot of mathematics

May 20, 2026 Research Milestone

An OpenAI model has disproved a central conjecture in discrete geometry

[Read the proof ↗](#)

[Read the companion remarks ↗](#)

The unit distance conjecture was disproved. The original proof is 1.5 pages long. A companion paper with comments from nine mathematicians was also released.

Ten results (August 1, 2026)



Progress on ten important open problems: a non-sofic group, Connes's rigidity conjecture, CVP, the Ehrhart volume conjecture, etc.

MATHEMATICS

OPENAI

OpenAI Research Catalog

372 result families in 722 manuscripts

October 6, 2026

An overview of the [manuscript collection](#). Results are grouped by subject; entry numbers follow the catalog order and do not indicate a ranking. PDF links follow each summary.

Contents

... a lot.

MATHEMATICS

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Contents

... a lot. The number has decreased a little — three papers on Hodge were retracted due to sign errors found by Astra.

(Counter)examples

- a counterexample to Grothendieck's conjecture
- a counterexample to the Jacobian conjecture
- an elliptic curve of rank ≥ 31 over $\mathbb{Q}$
- a Hadamard matrix of order 668

All the examples can be verified easily.¹ In this talk, we are not going to discuss these examples.

¹The counterexample to Grothendieck's conjecture was formalized by Akhil Mathew with AI. The other three (counter)examples are also formalized in some form; the elliptic curve's rank has been formally verified by Bhavik Mehta.

Understanding AI-generated proofs

Question

We have a lot of new results from non-humans. The write-ups are often “slop” and not easily readable. How can we understand them?

Understanding AI-generated proofs

Here are some very high-level tips.

Tip

- Use AI to read AI proofs. Otherwise, you need to talk to the alien yourself.
- You can do much more than prompting “please explain this proof in detail”.
- The existence of a formal proof does NOT guarantee the correctness of the informal proof.

Sphere packing problem and sign uncertainty principle

In this talk, I will describe my attempts to digest the proof — both informal and formal — of the following theorem.

Theorem (ChatGPT? OpenAI? OpenAI employee(s)?, 2026)

We have

$$\lim_{d \rightarrow \infty} \text{LP}_d^{1/d} = \sqrt{\frac{e}{2\pi}}$$

and

$$\lim_{d \rightarrow \infty} \frac{A_+(d)}{\sqrt{d}} = \lim_{d \rightarrow \infty} \frac{A_-(d)}{\sqrt{d}} = \frac{1}{\pi}.$$

Sphere packing problem and sign uncertainty principle

Question

Why this one?

- ❶ The optimal Cohn–Elkies exponent was on my TODO list.
- ❷ I worked on improving the upper bound for $A_+(d)$ in my thesis.²
- ❸ I have no expertise in the other problems.

²When the result was announced, I had to rush (with AI) to finish the remaining technical part, on which I had made no progress for months. Although my result is asymptotically worse than OpenAI's, the bound is explicit and the proof is completely different. Anyway, I was very satisfied with ChatGPT's argument.

Informal proof

Sphere packing problem

Question

For a given $d \geq 1$, find the densest sphere (in fact, ball) packing of $\mathbb{R}^d$ and its density Δ_d .

- $d = 1$ (Trivial)



$$\bigcup_{n \in \mathbb{Z}} [2n - 1, 2n + 1]$$

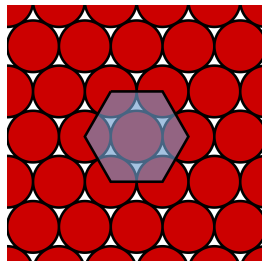
$$\Delta_1 = 1$$

Sphere packing problem

Question

For a given $d \geq 1$, find the densest sphere (in fact, ball) packing of $\mathbb{R}^d$ and its density Δ_d .

- $d = 1$ (Trivial)
- $d = 2$ (Tóth, 1940) Hexagonal packing



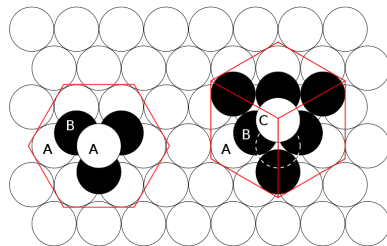
$$\Delta_2 = \frac{\pi}{2\sqrt{3}}$$

Sphere packing problem

Question

For a given $d \geq 1$, find the densest sphere (in fact, ball) packing of $\mathbb{R}^d$ and its density Δ_d .

- $d = 1$ (Trivial)
- $d = 2$ (Tóth, 1940) Hexagonal packing
- $d = 3$ (Hales, 1998) Cannonball packing



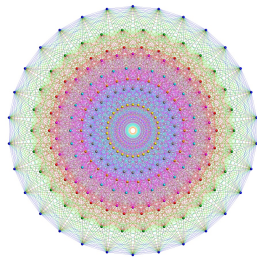
$$\Delta_3 = \frac{\pi}{3\sqrt{2}}$$

Sphere packing problem

Question

For a given $d \geq 1$, find the densest sphere (in fact, ball) packing of $\mathbb{R}^d$ and its density Δ_d .

- $d = 1$ (Trivial)
- $d = 2$ (Tóth, 1940) Hexagonal packing
- $d = 3$ (Hales, 1998) Cannonball packing
- $d = 8$ (Viazovska, 2016) E_8 lattice packing



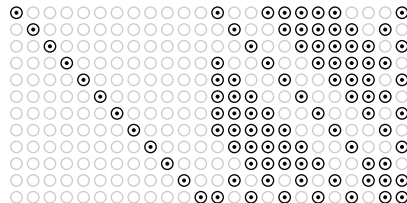
$$\Delta_8 = \frac{\pi^4}{384}$$

Sphere packing problem

Question

For a given $d \geq 1$, find the densest sphere (in fact, ball) packing of $\mathbb{R}^d$ and its density Δ_d .

- $d = 1$ (Trivial)
- $d = 2$ (Tóth, 1940) Hexagonal packing
- $d = 3$ (Hales, 1998) Cannonball packing
- $d = 8$ (Viazovska, 2016) E_8 lattice packing
- $d = 24$ (Cohn–Kumar–Miller–Radchenko–Viazovska, 2016) Leech lattice packing



All other dimensions are open.

$$\Delta_{24} = \frac{\pi^{12}}{12!}$$

Cohn–Elkies linear programming bound

Theorem (Cohn–Elkies, 2003)

Let $r > 0$. Assume that there exists a nice function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying

- ❶ $f(\mathbf{0}) = \hat{f}(\mathbf{0}) > 0$,
- ❷ $f(\mathbf{x}) \leq 0$ for all $\|\mathbf{x}\| \geq r$,
- ❸ $\hat{f}(\mathbf{y}) \geq 0$ for all $\mathbf{y} \in \mathbb{R}^d$,

where $\hat{f}$ is the Fourier transform of f . Then

$$\Delta_d \leq \text{vol}(B_{r/2}^d) = \left(\frac{r}{2}\right)^d \frac{\pi^{d/2}}{(d/2)!}.$$

Define LP_d as the infimum of the right-hand side. Viazovska and CKMRV constructed “magic functions” for $d = 8, 24$, proving the optimality of the E_8 and Leech lattice packings.

Theorem (Kabatiansky–Levenshtein, 1978)

$$\Delta_d \leq 2^{-(0.5990\dots+o(1))d}, \quad d \rightarrow \infty.$$

- Cohn–Zhao (2014) proved that the LP bound is at least as strong as the KL bound.
- The exponent $0.5990\dots$ has not moved since 1978. Only the prefactor was improved (Cohn–Zhao 2014, Sardari–Zargar 2024, Zargar 2024).
- The best lower bound is $\Delta_d \geq cd^2 \log \log d 2^{-d}$ (Gargava–Gröbner–Simkin, 2026).

Conjecture (Afkhami–Jeddi–Cohn–Hartman–de Laat–Tajdini, 2020)

$$\lim_{d \rightarrow \infty} \text{LP}_d^{1/d} = \sqrt{\frac{e}{2\pi}}, \quad \text{and} \quad \Delta_d \leq 2^{-(\frac{1}{2} \log_2(2\pi/e) + o(1))d} = 2^{-(0.6044\dots+o(1))d}$$

Sign uncertainty principle (Bourgain–Clozel–Kahane, 2010)

Question

For each dimension $d \geq 1$, find the smallest real number $\rho := A_+(d) \geq 0$ for which there exists a nonzero, radial, integrable function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ such that

- ❶ $\hat{f} = f$
- ❷ $f(\mathbf{0}) \leq 0$
- ❸ $f(\mathbf{x}) \geq 0$ whenever $\|\mathbf{x}\| \geq \rho$.

Uncertainty principle: $A_+(d) > 0$.

There's also a (-1) -version of the problem, where we require $\hat{f} = -f$ and $f(\mathbf{0}) = 0$. The corresponding optimal value is denoted by $A_-(d)$.

The only cases where the exact values of $A_{\pm}(d)$ are known are $A_+(12) = \sqrt{2}$ (Cohn–Gonçalves) and $A_-(1) = 1$, $A_-(2) = (4/3)^{1/4}$, $A_-(8) = \sqrt{2}$, $A_-(24) = 2$.

Known bounds for general d

Theorem (Bourgain–Clozel–Kahane, 2010)

$$0.2420 \approx \frac{1}{\sqrt{2\pi e}} < \frac{A_+(d)}{\sqrt{d}} < \frac{1}{\sqrt{2\pi}} + o(1) \approx 0.3989.$$

Theorem (L. 2026)

For every multiple d of 4, we have

$$A_+(d) \leq \sqrt{2 \left\lfloor \frac{d}{16} \right\rfloor} + 2.$$

Conjecture (Afkhami–Jeddi–Cohn–Hartman–de Laat–Tajdini, 2020)

$$\lim_{d \rightarrow \infty} \frac{A_+(d)}{\sqrt{d}} = \lim_{d \rightarrow \infty} \frac{A_-(d)}{\sqrt{d}} = \frac{1}{\pi}.$$

Main result

Theorem (ChatGPT? OpenAI? OpenAI employee(s)?, 2026)

We have

$$\lim_{d \rightarrow \infty} \text{LP}_d^{1/d} = \sqrt{\frac{e}{2\pi}}$$

and

$$\lim_{d \rightarrow \infty} \frac{A_+(d)}{\sqrt{d}} = \lim_{d \rightarrow \infty} \frac{A_-(d)}{\sqrt{d}} = \frac{1}{\pi}.$$

In other words, both conjectures of Afkhami-Jeddi-Cohn-Hartman-de Laat-Tajdini are true.

High-level comments

- In my opinion, the **Mellin transform** is a key idea. Note that Cohn–Miller (2016) studied the Mellin transforms of (hypothetical) magic functions.
- **Lower bound:** The proportion of mass of g in the ball of radius $c\sqrt{d}$ for $c < 1/\pi$ is $o(1)$, by relating it to the Mellin transform of g and estimating it on a horizontal strip using the Poisson inequality.
- **Upper bound:** Start from the Mellin transform of a Gaussian, and then perturb it to get a function with the desired properties with the last sign change radius $(1/\pi + o(1))\sqrt{d}$.
- The proof of the lower bound is much simpler than that of the upper bound.
- The result is **asymptotic** — it does not give optimal functions or bounds for any fixed d .

- The Codex app, running locally on my MacBook in the directory for my personal homepage, which contains my blog posts.
- The corresponding blog posts³ were also written with the help of ChatGPT, and I rewrote most of the drafts.

Tip

- I added a temporary directory for “experiments” where ChatGPT can freely write scripts and draw some figures for me.
- Download relevant papers locally; it seems easier for AI to read local files than to query the web every time.

³Part 1 (informal) and Part 2 (formal).

Lower bound

Lower bound

To get the lower bound of $1/\pi$ for the sign uncertainty principle, we will prove:

Proposition (Proposition 3.7)

For every $0 < c < 1/\pi$ and all large d : there is no nonzero $g \in L^1(\mathbb{R}^d)$ with $\hat{g} = \pm g$, $g(\mathbf{0}) = 0$ and $g(\mathbf{x}) \geq 0$ for $\|\mathbf{x}\| \geq c\sqrt{d}$.

We may assume g is radial (average over rotations) and Schwartz (mollifier argument).

We have $\int g = \hat{g}(\mathbf{0}) = 0$, so the negative part $g_- = \max\{-g, 0\}$ carries exactly half of $\|g\|_1$, and all of it sits inside the ball:

$$\frac{\|g\|_1}{2} = \int_{\mathbb{R}^d} g_-(\mathbf{x}) \, d\mathbf{x} = \int_{\|\mathbf{x}\| < c\sqrt{d}} g_-(\mathbf{x}) \, d\mathbf{x} \leq \int_{\|\mathbf{x}\| < c\sqrt{d}} |g(\mathbf{x})| \, d\mathbf{x}.$$

So it suffices to show that the mass of g in the ball of radius $c\sqrt{d}$ is $o(\|g\|_1)$. This observation is not new; it was already used in Bourgain–Clozel–Kahane.

Lower bound

They proved a stronger statement:

Proposition (Proposition 3.1)

For every $0 < c < 1/\pi$, there exist $C_c, \gamma_c > 0$ and $d_0(c) \in \mathbb{N}$ such that, for every $d \geq d_0(c)$, every $\varsigma \in \{-1, +1\}$, and every nonzero $g \in \mathcal{S}_{\text{rad}}(\mathbb{R}^d, \mathbb{R})$ satisfying $\hat{g} = \varsigma g$ and $g(0) = 0$,

$$\int_{\|\mathbf{x}\| < c\sqrt{d}} |g(\mathbf{x})| d\mathbf{x} \leq C_c e^{-\gamma_c d} \|g\|_1.$$

Tip

You can try to come up with a simplified proof. For example, can we prove the weaker statement that $\int_{\|\mathbf{x}\| < c\sqrt{d}} |g(\mathbf{x})| d\mathbf{x} = o(\|g\|_1)$ with a simpler argument?

If f is admissible with $f(\mathbf{x}) \leq 0$ for $\|\mathbf{x}\| \geq r$, then by considering $g = \hat{f} - f$, we get

$$r \geq A_-(d) \geq \left(\frac{1}{\pi} + o(1) \right) \sqrt{d}.$$

Hence

$$\text{LP}_d^{1/d} \geq \frac{r}{2} \text{vol}(B_1^d)^{1/d} \geq \left(\frac{1}{\pi} - o(1) \right) \cdot \frac{1}{2} \cdot \sqrt{d} \text{vol}(B_1^d)^{1/d} \longrightarrow \frac{1}{2} \cdot \frac{1}{\pi} \cdot \sqrt{2\pi e} = \sqrt{\frac{e}{2\pi}},$$

where $\sqrt{d} \text{vol}(B_1^d)^{1/d} \rightarrow \sqrt{2\pi e}$ by Stirling's formula. The mysterious-looking constant is just $1/\pi$ in disguise, so we can focus on the sign uncertainty principle and $1/\pi$.

Mellin transform

Let $\lambda := d/2$ and $S_d := 2\pi^{d/2}/\Gamma(d/2)$ (area of the unit sphere). For radial $g = g(r)$, define

$$M_g(z) = \int_0^\infty g(r)r^{z-1}dr, \quad X_g(t) := M_g(\lambda - it), \quad g(r) = \frac{r^{-\lambda}}{2\pi} \int_{\mathbb{R}} X_g(t)r^{it}dt.$$

Let $R = c\sqrt{d}$. Define a normalized Mellin transform $Z_g(t)$ as

$$Z_g(t) = \frac{S_d}{\|g\|_1} R^{\lambda+it} X_g(t)$$

Then, for any $-1 < \sigma < 1$,

$$\frac{1}{\|g\|_1} \int_{\|\mathbf{x}\| < R} |g(\mathbf{x})|d\mathbf{x} \leq \frac{1}{2\pi(1-\sigma)\lambda} \int_{\mathbb{R}} |Z_g(s + i\sigma\lambda)|ds,$$

and the rest of the proof is about bounding $|Z_g|$ on the horizontal strip $|\Im t| < \lambda$. $\sigma = \sigma(c)$ will be chosen later.

Where does $1/\pi$ come from?

I don't have a good explanation! It appears in the proof as follows: Let $\theta = \pi(1 + \sigma)/2$ and $P_\sigma(T)$ be the Poisson kernel on the strip,

$$P_\sigma(T) = \frac{\sin \theta}{4(\cosh(\pi T/2) - \cos \theta)}.$$

Let $M_\sigma = \int_{\mathbb{R}} P_\sigma(T) dT = (1 - \sigma)/2$. Lemmas 3.2 and 3.3 prove

$$\log |Z(s + i\sigma\lambda)| \leq \lambda M_\sigma (\log(2\pi c^2) + J_\sigma) + O_\sigma(\log(2 + \lambda))$$

where

$$J_\sigma = -\frac{1}{M_\sigma} \int_{\mathbb{R}} P_\sigma(T) \int_0^1 \log \sqrt{x^2 + \frac{T^2}{4}} dx dT, \quad \lim_{\sigma \rightarrow 1^-} J_\sigma = \log \frac{\pi}{2}.$$

Then $1/\pi$ is a threshold for c such that $\log(2\pi c^2) + J_\sigma < 0$ for some $\sigma = \sigma(c) < 1$.

Upper bound

Upper bound

We need to construct functions $f = f_d$ with $r(f) \approx \sqrt{d}/\pi$ as $d \rightarrow \infty$.

We will construct three Schwartz functions f_+, f_-, f_0 with $R = R_{\epsilon, d} > 0$ such that

$$\hat{f}_- = f_+, \quad \hat{f}_0 = f_0, \quad f_-(0) = f_+(0) > 0, \quad f_0(0) = 0,$$

and

$$f_-(\mathbf{x}) < 0 < f_0(\mathbf{x}) \text{ for } \|\mathbf{x}\| \geq R, \quad f_+(\mathbf{x}) > 0 \quad \forall \mathbf{x} \in \mathbb{R}^d.$$

Also,

$$\lim_{\epsilon \rightarrow 0^+} \lim_{d \rightarrow \infty} \frac{R_{\epsilon, d}}{\sqrt{d}} = \frac{1}{\pi}.$$

Then the asymptotic upper bounds for $\text{LP}_d^{1/d}$, $A_-(d)$, and $A_+(d)$ follow by considering f_- , $f_+ - f_-$, and f_0 , respectively.

Construction of f_j

Start with the Gaussian and its Mellin transform

$$g_G(r) = 2\pi^{\lambda/2}e^{-\pi r^2}, \quad X_g(t) = \pi^{it/2}\Gamma\left(\frac{\lambda - it}{2}\right)$$

Multiply by two additional factors: for $j \in \{+, -, 0\}$,

$$X_{f_j}(t) = X_g(t) \cdot P_j\left(\frac{t}{\lambda}\right) \cdot \exp\left(\lambda h\left(\frac{t}{\lambda}\right)\right), \quad f_j(r) = \frac{r^{-\lambda}}{2\pi} \int_{\mathbb{R}} X_{f_j}(t) r^{it} dt$$

where

- P_j is a low-degree polynomial controlling the sign of $f_j(r)$.
- h is a “perturbation” term of the form

$$h(\zeta) = \int_0^\infty w(a)(\cos(a\zeta) - 1) da$$

controlling the last sign change radius of $f_j(r)$.

For $\beta > 0$, the proof chooses

$$P_+(\zeta) = 1 + \zeta^2 + \beta + i\zeta(1 + \zeta^2)$$

$$P_-(\zeta) = 1 + \zeta^2 + \beta - i\zeta(1 + \zeta^2)$$

$$P_0(\zeta) = -(1 + \zeta^2)$$

Why?

Choice of P_j

There are several properties that P_j should satisfy:

- $P_-(-\zeta) = P_+(\zeta) \ (\Leftrightarrow \hat{f}_+ = f_-)$
- $P_0(-\zeta) = P_0(\zeta) \ (\Leftrightarrow \hat{f}_0 = f_0)$
- $P_-(-i) = P_+(-i) > 0, P_0(-i) = 0 \ (\Leftrightarrow f_-(0) = f_+(0) > 0, f_0(0) = 0)$
- $P_j(-x) = \overline{P_j(x)}$ for $x \in \mathbb{R} \ (\Leftrightarrow f_j \text{ is real-valued})$
- $P_-(iu) < 0 < P_0(iu), P_+(iu) > 0$ for certain ranges of $u \ (\Leftrightarrow f_-(r) < 0 < f_0(r), f_+(r) > 0 \text{ for certain ranges of } r)$

The above choice of P_j is the simplest one satisfying all the properties.

Tip

When AI gives you a construction that seems to come out of nowhere, ask it to explain what the characterizing properties of the construction are.

Without the perturbation, the Mellin inversions g_j of $X_{g_j}(t) := X_g(t)P_j(t/\lambda)$ are

$$g_+(r) = g_G(r) \left[\beta + \frac{8\pi r^2}{\lambda^3} (\pi^2 r^4 - (2\lambda + 3)\pi r^2 + (\lambda + 1)^2) \right]$$

$$g_-(r) = g_G(r) \left[\beta + \frac{8\pi r^2}{\lambda^3} (-\pi^2 r^4 + (\lambda + 3)\pi r^2 - (\lambda + 1)) \right]$$

$$g_0(r) = g_G(r) \frac{4\pi r^2(\pi r^2 - \lambda - 1)}{\lambda^2}$$

These functions give the $1/\sqrt{2\pi}$ upper bound of Bourgain–Clozel–Kahane. This is the best you can do with Gaussian $\times$ polynomial of degree $o(d)$, by Cohn–Dong–Gonçalves (2024).

Sign of f_j and P_j

We can write $f_j(r)$ in a logarithmic coordinate $r = e^v$ where $v = v(u)$ will be chosen later.

$$f_j(r) = \frac{\lambda E_\lambda(i\lambda u)}{2\pi} r^{-(1+u)\lambda} \int_{\mathbb{R}} e^{\mathcal{L}_u(T)} P_j(T + iu) dT$$

where

$$E_\lambda(t) = \pi^{it/2} \Gamma\left(\frac{\lambda - it}{2}\right) e^{\lambda h(t/\lambda)}, \quad \mathcal{L}_u(T) = \log \frac{E_\lambda(\lambda(T + iu))}{E_\lambda(i\lambda u)} + i\lambda T v.$$

We define $v = v(u)$ so that $T = 0$ is a critical point of $\mathcal{L}_u$.

- Lemma 4.8: the integral is dominated by $T \approx 0$, and the sign of $f_j(e^{v(u)})$ is the same as the sign of $P_j(iu)$ for large d and certain ranges of u .
- Lemma 4.10: $f_+(r)$ is approximately Gaussian near $r = 0$, so it is positive.

Last sign change radius

We have

$$v(u) = -\frac{1}{2} \log \pi + \frac{1}{2} \psi \left(\frac{\lambda(1+u)}{2} \right) + ih'(iu).$$

Lemma 4.8 shows that, for sufficiently large d , the last sign change radius of f_- and f_0 is bounded by

$$R_{d,\epsilon} = e^{v(u_0)} = \frac{1}{\sqrt{\pi}} \exp \left(\frac{1}{2} \psi \left(\frac{\lambda(1+u_0)}{2} \right) \right) \cdot \exp(ih'(iu_0))$$

where $u_0 = 1 + \frac{\epsilon}{4} > 1$. By the digamma asymptotic, we have

$$\lim_{d \rightarrow \infty} \frac{R_{d,\epsilon}}{\sqrt{d}} = \sqrt{\frac{1+u_0}{4\pi}} \exp(ih'(iu_0))$$

Choice of h

With the ansatz

$$h(\zeta) = \int_0^\infty w(a)(\cos(a\zeta) - 1)da,$$

the limit becomes

$$\sqrt{\frac{1+u_0}{4\pi}} \exp\left(\int_0^\infty w(a)a \sinh(u_0 a) da\right).$$

But where the ansatz come from? The manuscript explains it as “dilation”, which is imprecise.⁴ I asked ChatGPT multiple times to explain the ansatz, and it succeeded once.

⁴Multiplying the Mellin transform by $\cos(at/\lambda) - 1$ replaces $g(r)$ by $\frac{e^a g(re^{a/\lambda}) + e^{-a} g(re^{-a/\lambda})}{2} - g(r)$, which is dilation. However, we are multiplying $e^{\lambda h(t/\lambda)}$, not $\lambda h(t/\lambda)$.

Choice of h

Prompt

Ok, let's get back to the question I already asked. My main question is, how the ansatz $h(\zeta) = \int_0^\infty w(a)(\cos(a\zeta) - 1)$ comes from. "Dilation" isn't much helpful here for me. I'm trying to write everything without assuming that h has the specific form, and try to "characterize" properties of h that needed to make desired argument works. You can find some of them in the part 1 markdown file. Now the radius, in terms of u -coordinate, is given by $\exp(v(u)) = 1/\sqrt{\pi} \cdot \exp(\psi(\lambda(1+u)/2)/2) \cdot \exp(ih'(iu))$, and the ratio $\exp(v(u))/\sqrt{d}$ converges to (by the asymptote of digamma function) $\sqrt{(1+u)/4\pi} \cdot \exp(ih'(iu))$. So one of the requirements would be $ih'(iu)$ to be a real number, i.e. $h'(iu)$ is purely imaginary.

But this still does not explain the ansatz for h . Also, to make radius smaller, we want $ih'(iu)$ to be small, which corresponds to making $w(a)$ small under ansatz. So what is the issue if $ih'(iu)$ or $w(a)$ is TOO small?

Choice of h

Response (ChatGPT)

(...)Therefore, if we choose

$$h(\zeta) = \int_0^\infty w(a)(\cos(a\zeta) - 1)da,$$

then

$$h(iu) - \Re h(T + iu) = \int_0^\infty w(a) \cosh(au)(1 - \cos(aT))da$$

The total damping becomes

$$D_u(T) = \int_0^\infty [\mu_{\lambda,1+u}(a) + \lambda w(a) \cosh(au)](1 - \cos(aT))da$$

That is the main advantage of the ansatz: a difficult inequality for a complex function on every horizontal contour becomes a comparison of real densities.(...)

Choice of h

In short, if you choose h of this form, then the comparison between h and D_u becomes a comparison between integrands, and w can be chosen to satisfy the desired inequality ($D_u(T) > 0$).

Tip

- We cannot figure out the model's exact CoT from its output (unless we have access to it). The best we can do is approximate it from the output.
- If you simply ask “why” or “how”, it often repeats what is already written in the manuscript. Ask explicitly for alternative explanations. It NEVER gives you the same answer for the same question twice.

Choice of w

A sufficient condition for $D_u(T) > 0$ is a nonnegative integrand, i.e.

$$w(a) \geq -\frac{e^{-(1+u)a}}{\lambda a(1 - e^{-2a/\lambda}) \cosh(au)} \xrightarrow{\lambda \rightarrow \infty, u \rightarrow 1} w_*(a) := -\frac{e^{-2a}}{2a^2 \cosh a}.$$

However, you cannot simply take $w = w_*$. Instead, we first truncate and slightly shrink it as

$$w_s(a) = b(a) w_*(a) \mathbf{1}_{[a_0, A]}(a), \quad b(a) = 1 - 2\epsilon(1 + a),$$

but it causes another issue, so we add a tiny positive bump

$$w_B(a) = \frac{Q}{\cosh a} \mathbf{1}_{[B, B+1]}(a)$$

and take $w = w_s + w_B$. Now choose parameters a_0, A, B, Q, β as functions of ϵ so that the proof works.

Where does $1/\pi$ come from?

Again, *I don't have a good explanation!* In the proof, it appears as

$$\frac{1}{\sqrt{2\pi}} \exp \left(\int_0^\infty w_*(a) a \sinh a da \right) = \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{1}{2} \int_0^\infty \frac{e^{-2a} \tanh a}{a} da \right) = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\frac{2}{\pi}} = \frac{1}{\pi}.$$

And *this* $1/\pi$ matches with *that* $1/\pi$ from the lower bound.

Tip

There is still a lot to understand. The existence of a proof of a statement is definitely not the end, but rather the beginning of understanding.

Formal proof

Why formalize?

There is more than one reason to formalize mathematics.

- Check the correctness of statements and proofs.
- Build a library of formal mathematics.
- Deepen our understanding of the proof.

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Formalization

How were the issues found? You need a single prompt, plus extra hours to check the output carefully.

Prompt

I'm trying to understand the proof of optimal exponent of Cohn-Elkies linear programming bound described in Chapter 1 of this paper: <https://cdn.openai.com/pdf/ten-proofs-oai.pdf>

In particular, I want you to figure out the correspondence between Lemmas/Statements in the pdf and the corresponding Lean code here:

<https://github.com/openai/ten-proofs/blob/main/SpherePacking.lean> Give me the list as a table.

Maybe one lemma correspond to multiple formalized statements in the Lean file.

ChatGPT and Claude flagged the same issues I mentioned.

Disclaimer. All the issues are “minor”, and you should NOT interpret the points as “the proof is wrong”. You can still formalize the “original proofs” (which we will see later), and the original argument is correct.

But I still want to emphasize that, if you want to check the correctness of a natural-language proof via formalization, you have to be very careful.

Tip

It is easy to double-check that something bad exists. It is harder to double-check that something good does not exist.

Missing results

Nothing about f_0 or $A_+(d)$ is formalized. You can guess this from the fact that the original formalization has `AntiFourierWitness` but no `SelfFourierWitness`.

```
structure AntiFourierWitness (d : ℕ) (R : ℝ) where
  function : TestFunction d
  real : IsRealValued function
  radial : IsRadial function
  nonzero : function ≠ 0
  anti_fourier : (ℱ function : TestFunction d) = -function
  zero_value : function (0 : Euclidean d) = 0
  eventually_nonneg :
    ∀ x : Euclidean d, R ≤ ‖x‖ → 0 ≤ (function x).re
```

Different proofs: Lemma 3.2

The original proof of the estimate of $|Z|$ uses the **Poisson inequality** for subharmonic functions on the upper half-plane (and a conformal map between the horizontal strip and the upper half-plane). However, you cannot find any subharmonic functions in the formalization, and in fact they do not exist in `mathlib` yet.

In fact, the formalization uses the **Phragmén–Lindelöf principle** on the strip.⁵ It is essentially equivalent to the Poisson inequality, but they are quite different in terms of the formalization.

⁵And `mathlib` *does* have a version of the Phragmén–Lindelöf principle on a horizontal strip. However, the formalization needs a slightly stronger version.

Different proofs: parameters

The definition of $h = h_\epsilon$ depends on several parameters $a_0, A, B, Q, \dots$, all of which depend on ϵ . The original manuscript and the formalization choose them differently:

	Manuscript	Lean
a_0	ϵ^2	ϵ^3
A	$\log(1/\epsilon)$	$10 \log(1/\epsilon)$
$b(a)$	$1 - 2\epsilon(1 + a)$	$1 - 10\epsilon(1 + a)$

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What if the parameter choices of the manuscript do not work?

Well, it is fine since we have a formal proof that is correct... isn't it?

I decided to refactor the Lean code. In particular,

- Make it more readable and modular. Add a blueprint.
- Prove every result in the report, including the missing theorems.
- Follow the original proofs (Poisson inequality, original choice of parameters, etc.).
- Figure out upstreamable results.

`https://github.com/seewoo5/cohn-elkies-refactor`

Reformalization

Almost all of the code was written by Claude, followed by a lot of interaction.

I first wrote a `RefactoringPlan.md` file by hand, which contains a list of references to look at (the original manuscript, the original formalization, drafts of my two blog posts, etc.) and some guidelines organized in three steps:

- 1 Step 1: Write a single `SpherePackingRefactored.lean` file. A list of things to address (e.g. missing results, use of `mathlib`, stylistic changes) is given.
- 2 Step 2: Split the file into `CohnElkies` and `CohnElkiesForMathlib` directories, the latter for upstreamable results.
- 3 Step 3: Make a blueprint using `verso-blueprint`.

Prompt

Follow `RefactoringPlan.md`.

When the initial refactoring was done, I made additional comments that were missing from the original plan, including

- Remove unnecessary results (e.g. approximate Cohn–Elkies exponent with 30-digit precision)
- Add new Comparator setup and add Theorem 1.2 to the challenge file
- Add missing results and fix imprecise statements
- Merge duplicated results (e.g. initially, there were `rotationalAverage` and `radialSymmetrizationAverage`, which are the same)
- More stylistic comments
- Add `formalization.yaml`, license file, etc.

Reformalization

Now we have both $+1$ and -1 eigenfunctions ($\mathbb{Z}^\times = \{\pm 1\}$):

```
structure SignEigenfunction (d : ℕ) (s : ℤ×) where
  toFun : Euclidean d → ℝ
  integrable : Integrable toFun
  fourier_eq : ∀ ξ : Euclidean d, ℱ (fun x ↦ (toFun x : ℂ)) ξ = ((s : ℤ) : ℂ) * toFun ξ
  ne_zero : toFun ≠ 0
  zero : toFun 0 = 0
```

L^1 to Schwartz reduction (TestFunction $d = \mathcal{S}(\mathbb{R}^d, \mathbb{C})$):

```
theorem exists_schwartz_approximation (hd : 0 < d) (h : SignEigenfunction d s) (hrad : IsRadial h) :
  ∃ q : ℕ → TestFunction d, (∀ n, IsRealValued (q n) ∧ IsRadial (q n) ∧
    (ℱ (q n) : TestFunction d) = ((s : ℤ) : ℂ) • q n ∧ q n 0 = 0) ∧
  Tendsto (fun n ↦ ∫ x, ‖q n x - (h x : ℂ)‖) atTop (ℵ 0) := by ...
```

Theorem 1.2 is complete:

```
def signRadius {d : ℕ} (g : Euclidean d → ℝ) : ℝ≥0∞ :=  
  ⌈ R : {R : ℝ≥0 // ∀ x : Euclidean d, (R : ℝ) ≤ ‖x‖ → 0 ≤ g x}, (R : ℝ≥0∞)  
  
def signUncertaintyConstant (ς : ℤx) (d : ℕ) : ℝ≥0∞ :=  
  ⌈ g : SignEigenfunction d ς, signRadius g.toFun  
  
theorem signUncertaintyConstant_div_sqrt_tendsto (ς : ℤx) :  
  Tendsto (fun d : ℕ ↦ signUncertaintyConstant ς d / ENNReal.ofReal (Real.sqrt d)) atTop  
    (λ (ENNReal.ofReal (Real.pi-1))) := by
```

Reformalization

The Poisson inequality for subharmonic functions, which is used to formalize the original proof:

```
structure SubharmonicOn (u :  $\mathbb{C} \rightarrow \text{EReal}$ ) (U : Set  $\mathbb{C}$ ) : Prop where
  upperSemicontinuousOn : UpperSemicontinuousOn u U
  ne_top :  $\forall z \in U, u\ z \neq \top$ 
  le_circleAverage :  $\forall z \in U, \forall r \text{ in } \mathcal{N}[>] (0 : \mathbb{R}), \forall a : \mathbb{R},$ 
     $u\ z \leq ((\text{circleAverage } (\text{fun } w \mapsto \text{EReal.truncateToReal } a\ (u\ w))\ z\ r : \mathbb{R}) : \text{EReal})$ 

noncomputable def poissonKernelHalfPlane (z :  $\mathbb{C}$ ) (x :  $\mathbb{R}$ ) :  $\mathbb{R} :=$ 
   $\pi^{-1} * z.\text{im} / ((x - z.\text{re})^2 + z.\text{im}^2)$ 

noncomputable def poissonIntegralHalfPlane (b :  $\mathbb{R} \rightarrow \mathbb{R}$ ) (z :  $\mathbb{C}$ ) :  $\mathbb{R} :=$ 
   $\int x, \text{poissonKernelHalfPlane } z\ x * b\ x$ 

theorem le_poissonIntegralHalfPlane {b :  $\mathbb{R} \rightarrow \mathbb{R}$ } {E : Finset  $\mathbb{R}$ } {M :  $\mathbb{R}$ }
  (hu : SubharmonicOn u {z |  $0 < z.\text{im}$ }) (hM :  $\forall z : \mathbb{C}, 0 < z.\text{im} \rightarrow u\ z \leq M$ )
  (hb : Integrable fun x  $\mapsto b\ x / (1 + x^2)$ ) (hbc :  $\forall x : \mathbb{R}, x \notin E \rightarrow \text{ContinuousAt } b\ x$ )
  (hbdry :  $\forall x : \mathbb{R}, x \notin E \rightarrow \limsup u\ (\mathcal{N}[\{z \mid 0 < z.\text{im}\}]\ (x : \mathbb{C})) \leq (b\ x : \text{EReal})$ ) :
   $\forall z : \mathbb{C}, 0 < z.\text{im} \rightarrow u\ z \leq (\text{poissonIntegralHalfPlane } b\ z : \text{EReal}) := \text{by ...}$ 
```

Proposition A.1 is formalized.

```
theorem exists_signRadius_eq_signUncertaintyConstant_neg_one (hd : 0 < d) :  
  ∃ g : SignEigenfunction d (-1),  
    signRadius (g : Euclidean d → ℝ) = signUncertaintyConstant (-1) d := by ...
```

```
theorem signUncertaintyConstant_one_lt_neg_one (hd : 0 < d) :  
  signUncertaintyConstant 1 d < signUncertaintyConstant (-1) d := by ...
```

Note that Claude initially gave up on the proof of the second theorem, due to technical difficulty.⁶ But after my “encouragement”, it was able to formalize it with a different argument.

⁶To show the strict inequality, one needs to prove that there exists an optimizer for the constant $A_-(d)$, which is proved in Cohn–Gonçalves using results from several other papers.

Reformalization

Prompt

I still prefer unconditional result, so let's try our best. So we are going to follow Cohn-Goncalves paper (<https://arxiv.org/abs/1712.04438>). I think you can start with Lemma 3.1 of the paper, which do not require the technical results.

Response (Claude)

(...) **Plan change (simpler than Amrein–Berthier).** The only quantitative input CG19 needs from Nazarov/Jaming is a uniform lower bound on the mass of normalized eigenfunctions outside a fixed ball. That follows from a compactness argument alone: if $\mathcal{F}f_n = cf_n$, $\|f_n\|_1 = 1$ and the mass outside B_R tends to 0, then (weak L^2 limit of $1_B f_n$ + pointwise convergence via the Fourier integral + dominated convergence/Fatou) the f_n converge pointwise to a continuous $G \neq 0$ supported in B_R with $\mathcal{F}G = cG$, contradicting the compact-support theorem we already have. No operator theory, no spectral theorem.(...)

Reformalization

Some fruitful byproducts for `mathlib`:

- If $g(\mathbf{x}) = f(A\mathbf{x})$ then $\hat{g}(\xi) = |\det A|^{-1} \hat{f}(A^{-T}\xi)$.
- If f is twice differentiable and $f, f', f'' \in L^1$, then $\hat{f} \in L^1$.
- $\text{Res}_{z=-n} \Gamma(z) = (-1)^n / n!$
- If f and $\hat{f}$ are compactly supported, then $f = 0$.
- Gauss's integral representation of the digamma function: $\psi(m) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-mt}}{1-e^{-t}} \right) dt$
- Subharmonic functions and Jensen's inequality ($\log |f|$ is subharmonic for holomorphic f).
- Poisson summation formula for lattices

Tip

When you have a formalization of an upstreamable result, search `mathlib` and try to generalize it further. This is a mathematical exercise, not just a Lean exercise.

Tip

- LLMs are smart enough to find alternative proofs that are easier to formalize, and this actually happens quite a lot. If your goal is to formalize a *specific* proof, then you need to check the output carefully.
- The best way to refactor formal code is NOT to use a bunch of SKILL.md files. It is for YOU to understand both the informal and the formal proof; then it becomes clear what is wrong with the formal proof and how to fix it.

What's next?

Tip

Some general tips on formalization:

- If you want to learn it but don't want to spend too much time, focus on learning how to *read*, rather than *write*.
- The best use case of autoformalization is to formalize your own work on the fly. In practice, you may not need to formalize everything from scratch. Your work may contain a complicated but formalizable argument, on which you can try AI. You still need to check the output yourself, but since it is YOUR work, it shouldn't be hard.

What's next

Now we have 700-ish more math results. It is possible that you were working on some of the problems in the catalog.⁷

Tip

- If you have decided to read the proof, and if you have spent a lot of time on the problem and already have some insights, you will be able to figure out the next possible moves.
 - The proof might be simplified.
 - The proof might be generalized to other problems.
 - The proof might be wrong.
- The reward for being the first to solve a problem seems to be decreasing. We can aim higher: e.g., with these new data points, let's see if we can get closer to developing a general theory that solves a class of problems at once.

⁷It happened to me again!

Tip

I think everyone should rethink what their main motivation for doing mathematics was.